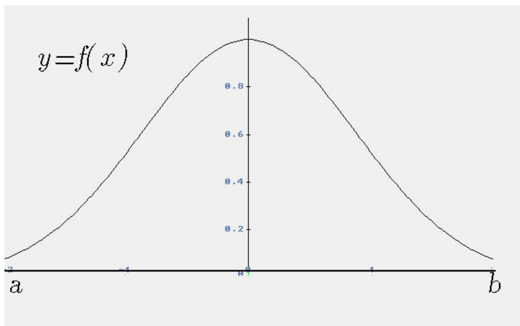
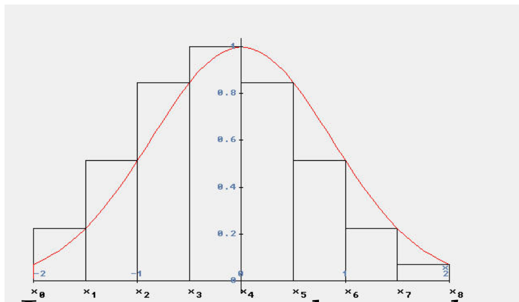
Approximate the area under a curve

$$y = f(x)$$

between two points $x = a$ and $x = b$

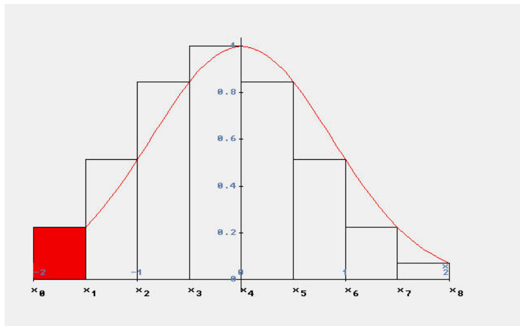


Divide the interval from a to b into subintervals. Approximate the area under the curve with the sum of the rectangular areas.



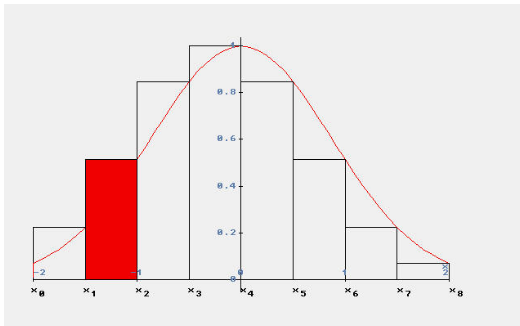
Let $x_0 = a$ and $x_8 = b$

The first rectangle has a height of $f(x_1)$ and a base of width Δx



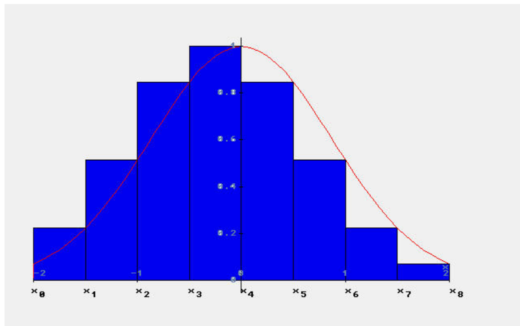
Area of 1st Rectangle $= f(x_1)\,\Delta x$

The second rectangle has a height of $f(x_2)$ and a base of width Δx



Area of 2nd Rectangle $= f(x_2)\,\Delta x$

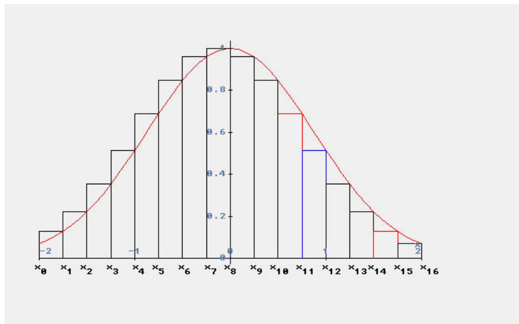
The sum of all the rectangular areas approximates the area under the curve



$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_8)\,\Delta x$$

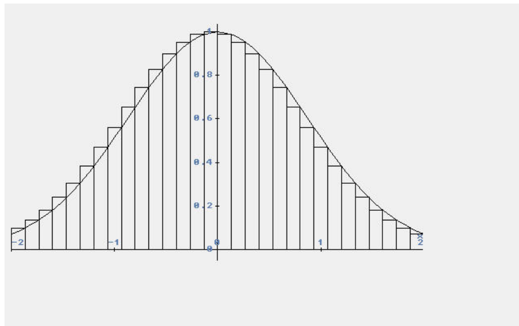
$$= \sum_{i=1}^{8} f(x_i)\,\Delta x$$

If more rectangles are used, the sum of the rectangular areas is even closer to the area under the curve



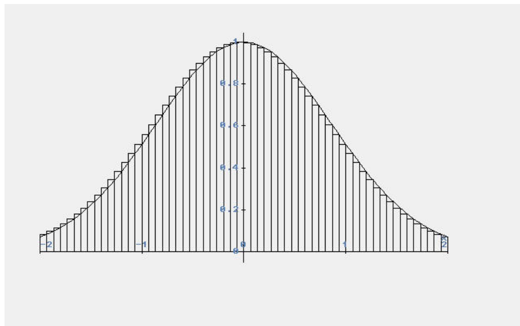
$$\sum_{i=1}^{16} f(x_i)\,\Delta x$$

The error in approximation approaches 0 as n gets larger



$$\sum_{i=1}^{30} f(x_i)\,\Delta x$$

The limit of the sum, as n approaches infinity, is the exact area under the curve



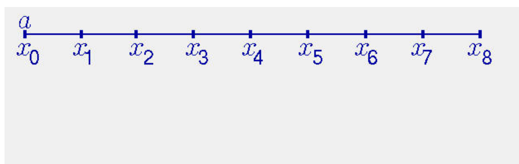
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

The limit of this sum is called the *definite integral* and is abbreviated by the following notation:

$$\int_a^b f(x)\,dx$$

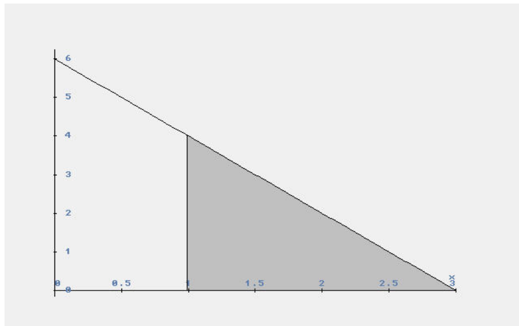
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

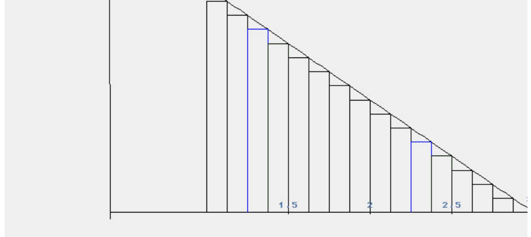
$$\Delta x = \frac{b-a}{n}$$



$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ x_3 &= a + 3\Delta x \\ &\vdots \\ x_i &= a + i\Delta x \end{aligned}$$

Find the area under $f(x) = 6 - 2x$ between $a = 1$ and $b = 3$

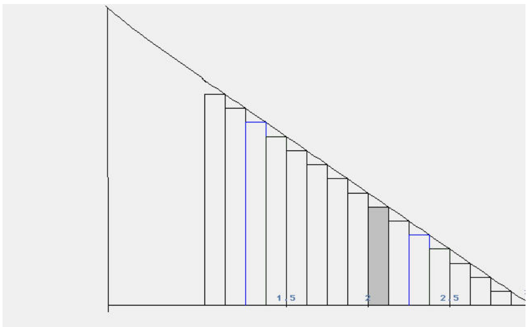




Divide the interval from $a = 1$ to $b = 3$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

Focus on the i^{th} rectangle



$$x_i = a + i\,\Delta x = 1 + i\,\frac{2}{n}$$

The height of the i^{th} rectangle is

$$f(x_i) = 6 - 2x_i = 6 - 2\left(1 + \frac{2i}{n}\right) = 4 - \frac{4i}{n}$$

The area of the i^{th} rectangle is

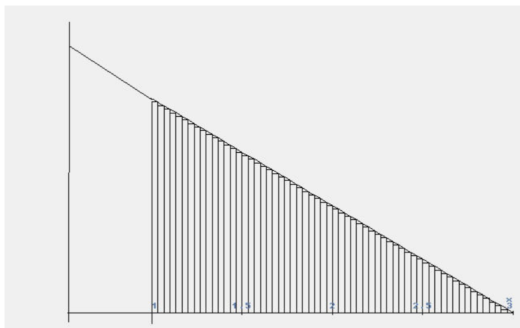
$$f(x_i)\,\Delta x = \left(4 - \frac{4i}{n}\right)\left(\frac{2}{n}\right) = \frac{8}{n} - \frac{8}{n^2}\,i$$

Sum all the areas of all n rectangles to form the Riemann sum.

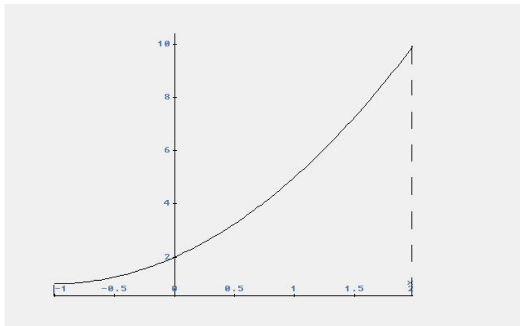
$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\,\Delta x &= \sum_{i=1}^{n}\left(\frac{8}{n} - \frac{8}{n^2}\,i\right)\\ &= \frac{8}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^2}\sum_{i=1}^{n} i\\ &= \frac{8}{n}\cdot n - \frac{8}{n^2}\cdot\frac{n(n+1)}{2}\\ &= 4 - \frac{4}{n}\end{aligned}$$

Finally, take the limit as $n \to \infty$

$$\int_1^3 (6 - 2x)\,dx = \lim_{n\to\infty} \left(4 - \frac{4}{n}\right) = 4$$



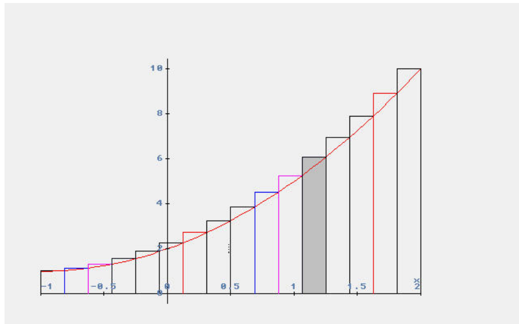
Find the area under $f(x) = x^2 + 2x + 2$ between $a = -1$ and $b = 2$



Divide the interval from $a = -1$ to $b = 2$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$$

Focus on the i^{th} rectangle



$$x_i = a + i\,\Delta x = -1 + i \cdot \frac{3}{n} = -1 + \frac{3i}{n}$$

The height of the i^{th} rectangle is

$$\begin{aligned} f(x_i) &= x_i^2 + 2x_i + 2 \\ &= \left(-1 + \frac{3i}{n}\right)^2 + 2\left(-1 + \frac{3i}{n}\right) + 2 \\ &= \frac{9}{n^2} i^2 + 1 \end{aligned}$$

The area of the i^{th} rectangle is

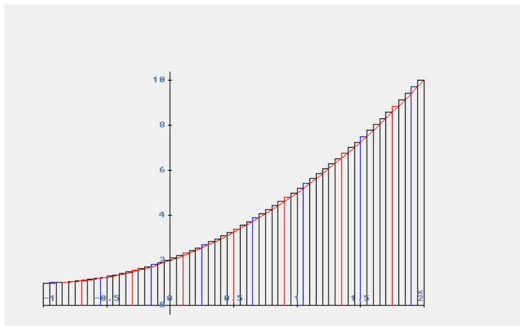
$$f(x_i)\,\Delta x = \left(\frac{9}{n^2} i^2 + 1\right)\left(\frac{3}{n}\right) = \frac{27}{n^3} i^2 + \frac{3}{n}$$

Sum the areas of all n rectangles

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\,\Delta x &= \sum_{i=1}^{n} \left(\frac{27}{n^3} i^2 + \frac{3}{n} \right) \\
&= \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot n \\
&= \frac{9}{2} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + 3
\end{aligned}$$

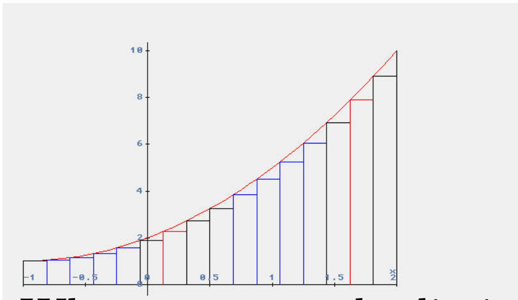
Finally, take the limit as $n \to \infty$

$$\lim_{n\to\infty}\left(\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+3\right)=9+3=12$$

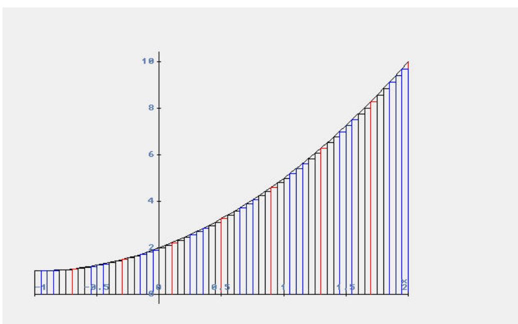


$$\int_{-1}^{2}\left(x^2+2x+2\right)\,dx=12$$

In these examples, we have chosen a point x_i on the right end-point of each subinterval. We could have chosen the left end-point of each subinterval instead.



When $n \to \infty$, the limit of the sum will still be the same as before



More generally, left c_i be *any* point in the i^{th} subinterval. This could be the left endpoint, the right endpoint, or any point in between. If $f(x)$ is continous on the interval $[a, b]$, the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

will be the same regardless of how the point c_i is selected.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$