MA 241 *Calculus I* **Fall 2016** **Exam III Solutions**

1. *(15 points)* If a cylindrical tank with base radius r meters has a volume of 54π cubic meters, then the surface area around this tank is given by: $S = 2\pi r^2 + \frac{108\pi}{r}$.
Find the value of r that minimizes the surface area.

$\frac{dS}{dr} = 4\pi r - \frac{108\pi}{r^2}$. This equals 0 when $4\pi r^3 = 108\pi$ so $r = 3$ meters.
Since $\frac{d^2S}{dr^2} = 4\pi + \frac{216\pi}{r^3} > 0$, we conclude that $r = 3$ meters minimizes the surface area.

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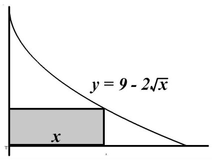
2. *(15 points)* A rectangular with base x meters is sitting on the x-axis with its left edge along the y-axis. The upper right corner of this rectangle lies along the curve $y = 9 - 2\sqrt{x}$.



Find the value of x that maximizes the area of this rectangle.

$$A = xy = 9x - 2x^{3/2} \qquad \frac{dA}{dx} = 9 - 3x^{1/2} \qquad \frac{d^2A}{dx^2} = -\frac{3}{2}x^{-1/2}$$

Since the second derivative is negative, the value of x that makes $\frac{dA}{dx} = 0$ will maximize the area A. The equation $\frac{dA}{dx} = 0$ implies $3\sqrt{x} = 9$ so $x = 9$ meters.

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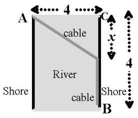
3. *(20 points)* A utility company must run a cable from point **A** to point **B** across a river. The river is 4 kilometers wide. Point **B** is 4 kilometers south of point **C** which is directly across the river from point **A**. The company will run the cable diagonally across the river to a point that is x kilometers south of point **C** and then the rest of the cable will be on land until it reaches point **B**. The cable costs 3 dollars per kilometer for the portion in the water and 1 dollar per kilometer for the portion on land. Find the value of x that minimizes the total cost.



If T denote the total cost then $T = (\text{cost of cable in water}) + (\text{cost of cable on land}) = 3\sqrt{16 + x^2} + 4 - x$

$$\frac{dT}{dx} = \frac{3x}{\sqrt{16 + x^2}} - 1 \quad \text{is zero when} \quad \frac{3x}{\sqrt{16 + x^2}} = 1$$

$$3x = \sqrt{16 + x^2} \quad \text{and, after squaring both sides, this becomes} \quad 9x^2 = 16 + x^2$$

$$x = \sqrt{2}$$

Since $\frac{d^2T}{dx^2} = \frac{48}{(16+x^2)^{3/2}}$ is positive for all $x > 0$, we may conclude that $x = \sqrt{2}$ km south of point **C** will minimize the total cost $T(x)$

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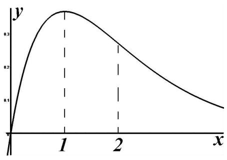
4. *(20 points)* Find all maximum, minimum and inflection points of $f(x) = xe^{-x}$ and use this information to sketch the curve.

$$f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x} \qquad f''(x) = (-1)e^{-x} - (1 - x)e^{-x} = (x - 2)e^{-x}$$

$f'(x) = 0$ at $x = 1$. Since $f''(1) < 0$, the point $\left(1,\ e^{-1}\right)$ must be a relative maximum point. $f''(x) = 0$ at $x = 2$ so the point $\left(2,\ 2e^{-2}\right)$ is a point of inflection.



5. The amount of money in a bank account after x years is given by the formula:

$$y = 100e^{0.01x}$$

Calculate the differential dy if x changes from 0 to 2.

$$dy = y'\,dx = e^{0.01x}\,dx = 2 \quad (\text{if } x = 0 \text{ and } dx = 2)$$

$\boxed{\textbf{a)}}$ 2 **b)** 200 **c)** $100e^{0.02}$ **d)** $100\left(e^{0.02} - 1\right)$

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6. Calculate the following sum:

$$\sum_{i=1}^{50}(4i-4) = 4\sum_{i=1}^{50} i - 4\sum_{i=1}^{50} 1 = 4 \cdot \frac{(50)(51)}{2} - (4)(50) = 4,900$$

$\boxed{\textbf{a)}}$ 4,900 **b)** 5,000 **c)** 5,096 **d)** 5,100

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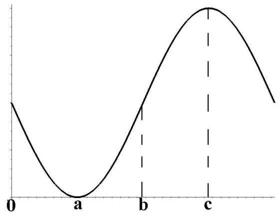
7. The picture below shows how the integral $\int_0^2 \frac{1}{x+1}\,dx$ can be approximated with a Riemann sum. In this picture, 6 rectangles have been used with the rightmost endpoint of each subinterval used to get the height of the rectangle

Which of the following expressions would be the correct Riemann sum?

a) $\sum_{i=1}^{6} \frac{1}{2i+1} \cdot \frac{i}{6}$ **b)** $\sum_{i=1}^{6} \frac{1}{\frac{i}{3}+1} \cdot \frac{1}{6}$ **c)** $\sum_{i=1}^{6} \frac{1}{i+1} \cdot \frac{1}{3}$ $\boxed{\textbf{d)}}$ $\sum_{i=1}^{6} \frac{1}{\frac{i}{3}+1} \cdot \frac{1}{3}$

Problems 8 - 10. All 3 problems refer to the function $y = F(x)$ that is sketched below. You are not given the formula for $F(x)$.



8. For $0 < x < a$, which one of the following is true?

a) $F'(x) < 0$ and $F''(x) < 0$ **b)** $F'(x) > 0$ and $F''(x) > 0$
$\boxed{\textbf{c)}}$ $F'(x) < 0$ and $F''(x) > 0$ **d)** $F'(x) > 0$ and $F''(x) < 0$

9. For $a < x < b$, which one of the following is true?

a) $F'(x) < 0$ and $F''(x) < 0$ $\boxed{\textbf{b)}}$ $F'(x) > 0$ and $F''(x) > 0$
c) $F'(x) < 0$ and $F''(x) > 0$ **d)** $F'(x) > 0$ and $F''(x) < 0$

10. For $b < x < c$, which one of the following is true?

a) $F'(x) < 0$ and $F''(x) < 0$ **b)** $F'(x) > 0$ and $F''(x) > 0$
c) $F'(x) < 0$ and $F''(x) > 0$ $\boxed{\textbf{d)}}$ $F'(x) > 0$ and $F''(x) < 0$