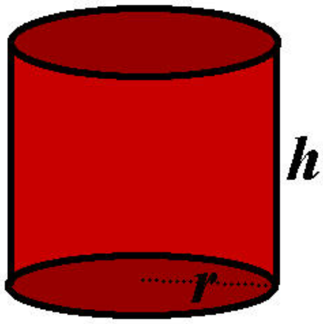
Minimum Cost Problem

The material on the top and bottom of a cylindrical container costs 3 cents per cm^2.

The material around the side costs 1 cent per cm^2.

The cylindrical container is supposed to have a volume of 480 cm^3. What are the dimensions of the cylinder that minimize the total cost? What is the minimum cost?



Let T be the total cost

$$\begin{aligned} T &= \begin{pmatrix} \text{Cost of} \\ \text{Bottom} \end{pmatrix} + \begin{pmatrix} \text{Cost of} \\ \text{Top} \end{pmatrix} + \begin{pmatrix} \text{Cost of} \\ \text{Side} \end{pmatrix} \\ &= 3 \cdot \pi r^2 + 3 \cdot \pi r^2 + 1 \cdot 2\pi r h \\ &= 6\pi r^2 + 2\pi r h \end{aligned}$$

The volume constraint $\pi r^2 h = 480$ implies that $h = \frac{480}{\pi r^2}$ and therefore,

$$T = 6\pi r^2 + 2\pi r \cdot \frac{480}{\pi r^2} = 6\pi r^2 + \frac{960}{r}$$

$$T = 6\pi r^2 + \frac{960}{r} = 6\pi r^2 + 960r^{-1}$$

$$\frac{dT}{dr} = 12\pi r - 960r^{-2} \qquad \frac{d^2T}{dr^2} = 12\pi + 1920r^{-3}$$

The second derivative $\frac{d^2T}{dr^2} = 12\pi + \frac{1920}{r^3}$ is positive, so the value of r that makes $\frac{dT}{dr}$ equal 0 must minimize the total cost.

$$\frac{dT}{dr} = 12\pi r - \frac{960}{r^2} = 0$$

$$12\pi r^3 - 960 = 0$$

$$r = \left(\frac{960}{12\pi}\right)^{1/3} = 2\left(\frac{10}{\pi}\right)^{1/3} \approx 2.94 \text{ cm}$$

$$h = \frac{480}{\pi r^2} = 12\left(\frac{10}{\pi}\right)^{1/3} \approx 17.7 \text{ cm}$$

$$T_{\text{min}} = 6\pi r^2 + 2\pi r h \approx \$4.90$$