MA 242 Review Exponential and Log Functions

Notes for today's class can be found at

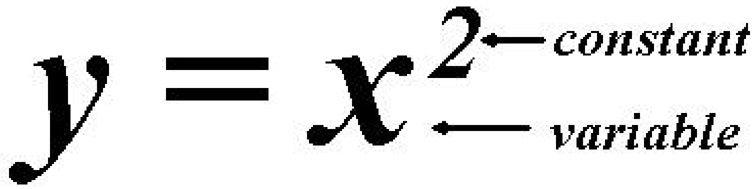
www.xecu.net/jacobs/index242.htm

$$\text{If } y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}$$

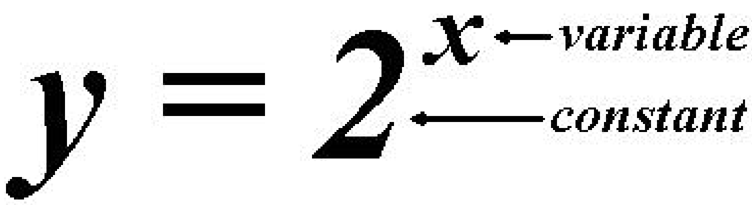
Example:

$$\text{If } y = x^2 \quad \text{then} \quad \frac{dy}{dx} = 2x^1 = 2x$$

Power Function



Exponential Function



Chromium-254 is a radioactive isotope with a half-life of approximately 1 month.

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Let's suppose we begin with an 8 gram sample of chromium.

After 1 month, we will be left with a 4 gram sample

Number of months	Mass of chromium
0	8 grams
1	4 grams
2	2 grams
3	1 gram
4	$\frac{1}{2}$ gram

Number of months	Mass of chromium
0	8 grams
1	$8 \cdot \frac{1}{2} = 4$ grams
2	$8 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2$ grams
3	$8 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$ gram
4	$8 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$ gram

Number of months	Mass of chromium
0	8 grams
1	$8 \cdot \frac{1}{2} = 4$ grams
2	$8 \cdot \left(\frac{1}{2}\right)^2 = 2$ grams
3	$8 \cdot \left(\frac{1}{2}\right)^3 = 1$ gram
4	$8 \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{2}$ gram

$$f(t) = 8 \cdot \left(\frac{1}{2}\right)^t$$

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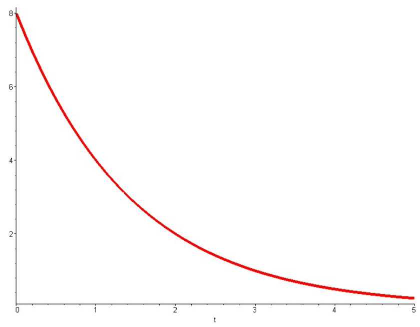

The number of giardia organisms will quadruple every day

Number of days	Number of organisms
0	1
1	4
2	$4^2 = 16$
3	$4^3 = 64$
4	$4^4 = 256$

$$f(t) = 4^t$$

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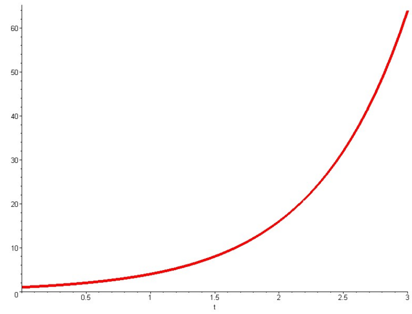

Derivative of a Power Function:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Derivative of an Exponential Function:

$$\frac{d}{dx}\left(a^x\right) = ????$$

$$f(x) = a^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$f(x) = a^x$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\
&= a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \cdot k
\end{aligned}$$

Derivative of a Power Function:

$$\frac{d}{dx}\left(x^{n}\right)=nx^{n-1}$$

Derivative of an Exponential Function:

$$\frac{d}{dx}\left(a^{x}\right)=k\cdot a^{x}$$

$$\frac{d}{dx}\left(a^x\right) = k \cdot a^x$$

$$\frac{d}{dx}\left(2^x\right) = (0.693)2^x$$

$$\frac{d}{dx}\left(3^x\right) = (1.099)3^x$$

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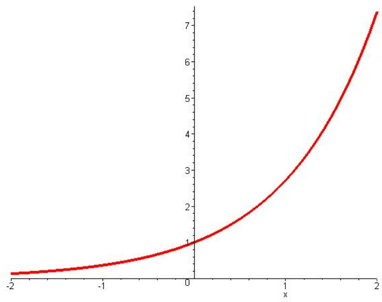
$$\frac{d}{dx}\left(3^x\right) = (1.099)3^x$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$e = 2.71828\ldots$$

$$y = e^x \qquad\qquad \frac{dy}{dx} = e^x$$



Problem: If $y = e^{\sin x}$, find $\frac{dy}{dx}$

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Let $u = \sin x$ so $y = e^u$. According to the *Chain Rule*,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx} = e^{\sin x} \cos x$$

Generalization:

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$$

Properties of exponential

1. $e^0 = 1$

2. $e^1 = e$

3. $e^{u+v} = e^u e^v$

4. $e^{u-v} = \frac{e^u}{e^v}$

5. $(e^u)^n = e^{nu}$

6. $\frac{d}{dx}(e^x) = e^x$

7. $\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$

Let $x > 0$

$$y = \log_b x \qquad \text{means } b^y = x$$

$$\ln x \text{ means } \log_e x$$

$$\text{so } y = \ln x \text{ implies } e^y = x$$

Properties of exponential and logarithms

1.	$e^0 = 1$	$\ln 1 = 0$
2.	$e^1 = e$	$\ln e = 1$
3.	$e^{u+v} = e^u e^v$	$\ln(xy) = \ln x + \ln y$
4.	$e^{u-v} = \frac{e^u}{e^v}$	$\ln \frac{x}{y} = \ln x - \ln y$
5.	$\left(e^u\right)^n = e^{nu}$	$\ln\left(x^n\right) = n \ln x$
6.	$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$

If $x > 0$ then:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int \frac{1}{x}\, dx = \ln x + C$$

Derivative Example:

$$\text{Let} \quad y = \sin(\ln x) \qquad \text{Find } \frac{dy}{dx}$$

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Let $u = \ln x$ so $y = \sin u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot \frac{1}{x} = \frac{1}{x}\cos(\ln x)$$

Similar Derivative Example:

$$\text{Let} \quad y = \ln(\sin x) \qquad \text{Find } \frac{dy}{dx}$$

$$\text{Let} \quad y = \ln(\sin x) \qquad \text{Find } \frac{dy}{dx}$$

Let $u = \sin x$ so $y = \ln u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$$

More generally, suppose:

$$y = \ln f(x)$$

Find the derivative

$$y = \ln f(x)$$

Let $u = f(x)$ so $y = \ln u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)} \cdot f'(x)$$

Example:
Find the derivative

$$y = \ln\left(\sqrt{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x})$$

Find the derivative

$$y = \ln\left(\sqrt{x}\right)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x^{1/2}} \cdot \frac{d}{dx}\left(x^{1/2}\right)\\ &= x^{-1/2} \cdot \frac{1}{2}x^{-1/2}\\ &= \frac{1}{2}x^{-1}\\ &= \frac{1}{2x}\end{aligned}$$

Example:
Find the derivative

$$y = \ln\left(\sqrt{x}\right)$$

Alternate solution:

$$y = \ln\left(\sqrt{x}\right) = \ln\left(x^{1/2}\right) = \frac{1}{2}\ln x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\ln x\right) = \frac{1}{2}\frac{d}{dx}(\ln x) = \frac{1}{2}\cdot\frac{1}{x} = \frac{1}{2x}$$

Find the derivative of:

$$y = \ln\left(x^2 \cos x\right)$$

$$y = \ln\left(x^2 \cos x\right) = \ln\left(x^2\right) + \ln\cos x = 2\ln x + \ln\cos x$$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{x} + \frac{1}{\cos x} \cdot (-\sin x) = \frac{2}{x} - \tan x$$

Find the derivative of:

$$y = \ln(\sec x + \tan x)$$

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$$y = \ln(\sec x + \tan x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

Find the derivative of:

$$y = \ln(\sec x + \tan x)$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x) \\
&= \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x) \\
&= \frac{1}{\sec x + \tan x} \cdot (\tan x + \sec x) \cdot \sec x \\
&= \sec x
\end{aligned}$$

$$\frac{d}{dx}(\ln(\sec x + \tan x)) = \sec x$$

$$\int \sec x \, dx = \ln(\sec x + \tan x) + C$$

Similarly

$$\frac{d}{dx}(\ln(\csc x + \cot x)) = -\csc x$$

$$\int \csc x \, dx = -\ln(\csc x + \cot x) + C$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$$

More generally,

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x} \qquad \frac{d}{dx}(\ln |f(x)|) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\frac{d}{dx}(\ln|f(x)|) = \frac{f'(x)}{f(x)} \qquad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \frac{1}{2x+1}\ dx$$

Let $u = 2x + 1$ $\qquad \frac{du}{dx} = 2$ $\qquad \frac{1}{2}du = dx$

$$\begin{aligned} \int \frac{1}{2x+1}\ dx &= \int \frac{1}{u} \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int \frac{1}{u}\ du \\ &= \frac{1}{2} \ln|u| + C \\ &= \frac{1}{2} \ln|2x+1| + C \end{aligned}$$

Alternate approach: Make use of the formula:

$$\int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)| + C$$

$$\int \frac{1}{2x+1}\ dx = \frac{1}{2}\int \frac{2}{2x+1}\ dx = \frac{1}{2}\ln|2x+1| + C$$

$$\int_0^{1/2} \frac{x}{2x+1}\, dx$$

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Let $u = 2x + 1$ $\qquad x = \frac{1}{2}(u-1)$ $\qquad dx = \frac{1}{2}\,du$

$$\begin{aligned}\int_0^{1/2} \frac{x}{2x+1}\,dx &= \int_1^2 \frac{\frac{1}{2}(u-1)\cdot\frac{1}{2}\,du}{u} \\ &= \frac{1}{4}\int_1^2 \left(1-\frac{1}{u}\right)\,du \\ &= \frac{1-\ln 2}{4}\end{aligned}$$

$$\int \tan x \ dx$$

$$\int \tan x \; dx = \int \frac{\sin x}{\cos x} \; dx$$

Let $t = \cos x \qquad dt = -\sin x \ dx \qquad -dt = \sin x \ dx$

$$\begin{aligned}
\int \tan x \ dx &= \int \frac{\sin x}{\cos x} \ dx \\
&= -\int \frac{1}{t} \ dt \\
&= -\ln|t| + C \\
&= -\ln|\cos x| + C
\end{aligned}$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$$

Let $t = e^x + e^{-x}$ $dt = \left(e^x - e^{-x}\right)\, dx$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \int \frac{1}{t}\, dt = \ln t + C = \ln\left(e^x + e^{-x}\right) + C$$

Note: $\frac{1}{2}\left(e^{x} - e^{-x}\right)$ is called a *hyperbolic sine*
and $\frac{1}{2}\left(e^{x} + e^{-x}\right)$ is called a *hyperbolic cosine*

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \qquad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \qquad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}\left(e^{x} - e^{-x}\right)}{\frac{1}{2}\left(e^{x} + e^{-x}\right)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\int \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\, dx = \int \tanh x \, dx$$