Limits

$$\lim_{t\to 0} \frac{(2+t)^2-4}{t}$$

$$\lim_{t\to 0} \frac{-t+\sqrt{t}}{t+\sqrt{t}}$$

$$\lim_{t\to 0} \frac{\tan 2t}{t}$$

$$\lim_{t\to 0} \frac{1-\cos t}{t}$$

Limits - all of the following are of the form $\frac{0}{0}$

$$\lim_{t\to 0} \frac{(2+t)^2-4}{t}$$

$$\lim_{t\to 0} \frac{-t+\sqrt{t}}{t+\sqrt{t}}$$

$$\lim_{t\to 0} \frac{\tan 2t}{t}$$

$$\lim_{t\to 0} \frac{1-\cos t}{t}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

This limit problem is said to be in the form $\frac{0}{0}$ if:

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

$$\lim_{x\to 0} \frac{x^3}{x} = \lim_{x\to 0} x^2 = 0$$

$$\lim_{x\to 0} \frac{x}{x^3} = \lim_{x\to 0} \frac{1}{x^2} = \infty$$

$$\lim_{x\to 0} \frac{7x^2}{2x^2} = \lim_{x\to 0} \frac{7}{2} = \frac{7}{2}$$

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$$\lim_{x \to 0} \frac{7x^2}{2x^2} = \lim_{x \to 0} \frac{7}{2} = \frac{7}{2}$$

Conclusion: Just knowing that a limit problem is in the form $\frac{0}{0}$ is not enough information to give you the answer.

$$\lim_{t \to 0} \frac{(2+t)^2 - 4}{t}$$

$$\begin{aligned}\lim_{t\to 0}\frac{(2+t)^2-4}{t} &= \lim_{t\to 0}\frac{4+4t+t^2-4}{t}\\ &= \lim_{t\to 0}\frac{4t+t^2}{t}\\ &= \lim_{t\to 0}(4+t)\\ &= 4\end{aligned}$$

$$\lim_{t \to 0} \frac{-t + \sqrt{t}}{t + \sqrt{t}}$$

$$
\begin{aligned}
\lim_{t\to 0} \frac{-t+\sqrt{t}}{t+\sqrt{t}} &= \lim_{t\to 0} \left(\frac{-t+\sqrt{t}}{t+\sqrt{t}}\right) \cdot \left(\frac{t-\sqrt{t}}{t-\sqrt{t}}\right) \\
&= \lim_{t\to 0} \frac{-t^2+2t\sqrt{t}-t}{t^2-t} \\
&= \lim_{t\to 0} \frac{-t+2\sqrt{t}-1}{t-1} \\
&= \frac{-0+2\sqrt{0}-1}{0-1} \\
&= 1
\end{aligned}
$$

$$\lim_{t \to 0} \frac{\tan 2t}{t}$$

Review: If θ is measured in *radians* then:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Therefore,

$$\lim_{t \to 0} \frac{\sin 2t}{2t} = 1$$

$$\begin{aligned}
\lim_{t\to 0} \frac{\tan 2t}{t} &= \lim_{t\to 0} \frac{\frac{\sin 2t}{\cos 2t}}{t} \\
&= \lim_{t\to 0} \frac{2}{\cos 2t} \cdot \frac{\sin 2t}{2t} \\
&= \frac{2}{\cos 0} \cdot 1 \\
&= 2
\end{aligned}$$

$$\lim_{t \to 0} \frac{1 - \cos t}{t}$$

$$\begin{aligned}
\lim_{t\to 0} \frac{1-\cos t}{t} &= \lim_{t\to 0} \left(\frac{1-\cos t}{t}\right) \cdot \left(\frac{1+\cos t}{1+\cos t}\right) \\
&= \lim_{t\to 0} \frac{1-\cos^2 t}{t(1+\cos t)} \\
&= \lim_{t\to 0} \frac{\sin^2 t}{t(1+\cos t)} \\
&= \lim_{t\to 0} \left(\frac{\sin t}{t}\right) \left(\frac{\sin t}{1+\cos t}\right) \\
&= 1 \cdot \frac{0}{1+1} \\
&= 0
\end{aligned}$$

L'Hôpital's Rule is a formula for calculating

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

that is true when:

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let $x = a + h$.
h approaching 0 is the same as x approaching a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Similarly,

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

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$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

If $f(a) = 0$ and $g(a) = 0$ then

$$f'(a) = \lim_{x \to a} \frac{f(x)}{x - a} \qquad g'(a) = \lim_{x \to a} \frac{g(x)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{f(x)}{x - a} \qquad g'(a) = \lim_{x \to a} \frac{g(x)}{x - a}$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

l'Hôpital's Rule:

If $f(x)$ and $g(x)$ are both approaching 0 as x approaches a then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{t \to 0} \frac{(2+t)^2 - 4}{t}$$

$$\lim_{t \to 0} \frac{(2+t)^2 - 4}{t} = \lim_{t \to 0} \frac{2(2+t)}{1} = \frac{2(2+0)}{1} = 4$$

$$\lim_{t \to 0} \frac{-t + \sqrt{t}}{t + \sqrt{t}}$$

$$
\begin{aligned}
\lim_{t\to 0}\frac{-t+\sqrt{t}}{t+\sqrt{t}} &= \lim_{t\to 0}\frac{-t+t^{1/2}}{t+t^{1/2}} \\
&= \lim_{t\to 0}\frac{-1+\frac{1}{2}t^{-1/2}}{1+\frac{1}{2}t^{-1/2}} \\
&= \lim_{t\to 0}\frac{-t^{1/2}+\frac{1}{2}}{t^{1/2}+\frac{1}{2}} \\
&= \frac{0+\frac{1}{2}}{0+\frac{1}{2}} \\
&= 1
\end{aligned}
$$

$$\lim_{t \to 0} \frac{\tan 2t}{t}$$

$$\lim_{t\to 0} \frac{\tan 2t}{t} = \lim_{t\to 0} \frac{2\sec^2 2t}{1} = 2\sec^2 0 = 2$$

$$\lim_{t \to 0} \frac{1 - \cos t}{t}$$

$$\lim_{t \to 0} \frac{1 - \cos t}{t} = \lim_{t \to 0} \frac{\sin t}{1} = \frac{\sin 0}{1} = 0$$

$$\lim_{x \to 0} \frac{\sin 3x}{4x}$$

$$\lim_{x\to 0}\frac{\sin 3x}{4x}=\lim_{x\to 0}\frac{3\cos 3x}{4}=\frac{3\cos 0}{4}=\frac{3}{4}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\left(x - \frac{\pi}{2}\right)^2}{\sin\left(\left(x - \frac{\pi}{2}\right)^2\right)}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\left(x - \frac{\pi}{2}\right)^2}{\sin\left(\left(x - \frac{\pi}{2}\right)^2\right)} = \lim_{x \to \frac{\pi}{2}} \frac{2\left(x - \frac{\pi}{2}\right)}{\cos\left(\left(x - \frac{\pi}{2}\right)^2\right) \cdot 2\left(x - \frac{\pi}{2}\right)}$$

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \frac{\left(x-\frac{\pi}{2}\right)^2}{\sin\left(\left(x-\frac{\pi}{2}\right)^2\right)} &= \lim_{x\to\frac{\pi}{2}} \frac{2\left(x-\frac{\pi}{2}\right)}{\cos\left(\left(x-\frac{\pi}{2}\right)^2\right)\cdot 2\left(x-\frac{\pi}{2}\right)} \\ &= \lim_{x\to\frac{\pi}{2}} \frac{1}{\cos\left(\left(x-\frac{\pi}{2}\right)^2\right)} \\ &= \frac{1}{\cos 0} = 1\end{aligned}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln(\sin x)}{(\pi - 2x)^2}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln(\sin x)}{(\pi - 2x)^2}$$

Note that the numerator is approaching $\ln\left(\sin \frac{\pi}{2}\right) = \ln 1 = 0$ and the denominator is approaching $\left(\pi - 2 \cdot \frac{\pi}{2}\right)^2 = (\pi - \pi)^2 = 0$ so l'Hôpital's Rule applies.

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln(\sin x)}{(\pi - 2x)^2} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cdot \cos x}{-4(\pi - 2x)}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln(\sin x)}{(\pi - 2x)^2} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cdot \cos x}{-4(\pi - 2x)}$$

Both numerator and denominator are approaching 0, so we can use l'Hôpital's Rule *again*.

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \frac{\ln(\sin x)}{(\pi-2x)^2} &= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}\cdot\cos x}{-4(\pi-2x)} \\
&= \left(\lim_{x\to\frac{\pi}{2}} \frac{1}{4\sin x}\right)\cdot\left(\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-(\pi-2x)}\right) \\
&= \frac{1}{4}\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{2x-\pi} \\
&= \frac{1}{4}\lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{2} \\
&= \left(\frac{1}{4}\right)\cdot\left(\frac{-1}{2}\right) = -\frac{1}{8}
\end{aligned}$$

$$\lim_{x \to 0} \frac{2^x - 5^x}{10x}$$

$$\lim_{x \to 0} \frac{2^x - 5^x}{10x}$$

Note that the numerator is approaching $2^0 - 5^0 = 1 - 1 = 0$
and the denominator is approaching $10 \cdot 0 = 0$
so l'Hôpital's Rule applies

$$y = 2^x$$

$$\ln y = \ln\left(2^x\right) = x \ln 2$$

$$\frac{1}{y}\,\frac{dy}{dx} = 1 \cdot \ln 2$$

$$\frac{dy}{dx} = y \ln 2 = 2^x \ln 2$$

$$\frac{d}{dx}\left(2^x\right) = 2^x \ln 2$$

Similarly,

$$\frac{d}{dx}\left(5^x\right) = 5^x \ln 5$$

$$\begin{aligned}
\lim_{x\to 0} \frac{2^x - 5^x}{10x} &= \lim_{x\to 0} \frac{2^x \ln 2 - 5^x \ln 5}{10} \\
&= \frac{2^0 \ln 2 - 5^0 \ln 5}{10} \\
&= \frac{\ln 2 - \ln 5}{10} \\
&= \frac{1}{10} \ln \frac{2}{5}
\end{aligned}$$