Notes on Centroids - 2D

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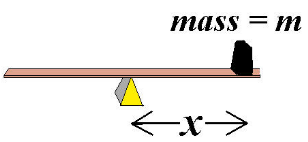
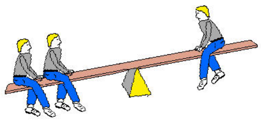
Review of Center of Mass in 1 Dimension

If an object of mass m kg is located x meters away from a fulcrum, then the quantity mx is the moment of the object.

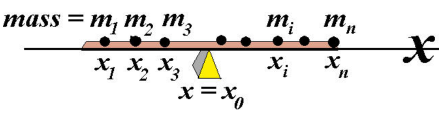


The moment measures the tendency to rotate around a particular point.



If there are several objects then the combined moment with respect to the point $x = x_0$ is:

$$\mathbf{M}_{x_0} = m_1(x_1 - x_0) + m_2(x_2 - x_0) + \cdots + m_n(x_n - x_0)$$



Or, in summation notation:

$$\mathbf{M}_{x_0} = \sum_{i=1}^{n} m_i(x_i - x_0)$$

If we omit the limits of the sum, we can write this in an even more abbreviated notation:

$$\mathbf{M}_{x_0} = \sum m_i(x_i - x_0)$$

The point $x = \overline{x}$ is the center of mass if the total moment is 0.

$$\mathbf{M}_{\overline{x}} = \sum m_i(x_i - \overline{x}) = 0$$

This simplifies to:

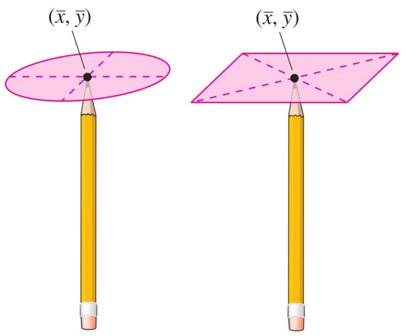
$$\sum m_i x_i - \overline{x} \sum m_i = 0$$

Therefore, the formula for the center of mass is:

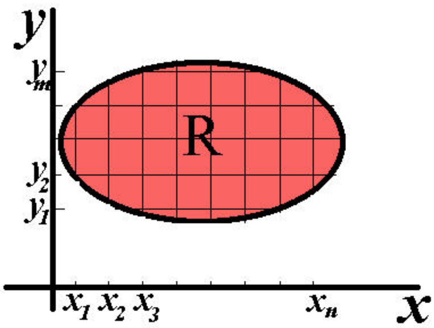
$$\overline{x} = \frac{\sum m_i x_i}{\sum m_i}$$

Center Of Mass in 2 Dimensions

Suppose that mass varies continuously in a two dimensional region R and the density at any point (x, y) (in kg/m^2) is given by $\delta = f(x, y)$. We wish to generalize the notion of center of mass so that at this point in region R, the region is balanced in all directions.

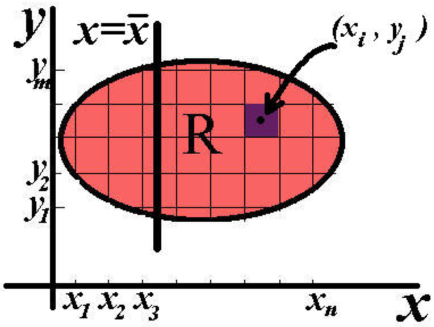


Let's begin by subdividing our region R into a grid.



Let's pick a particular section, say in row i and column j and let $(x_i,\ y_j)$ be some point in this region. If the region has area ΔA then the mass of this region is approximately:

$$f(x_i,\ y_j)\Delta A$$



The displacement of this point relative to the vertical line $x = \overline{x}$ is $x_i - \overline{x}$ so the approximate moment of this section with respect to the line $x = \overline{x}$ is:

$$(x_i - \overline{x})f(x_i, y_j)\Delta A$$

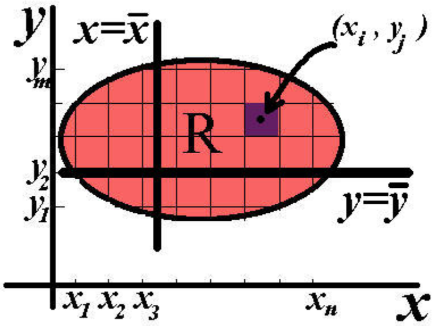
Add this up over the entire region R to approximate the total moment about this vertical line.

$$M_{\overline{x}} \approx \sum_i \sum_j (x_i - \overline{x})f(x_i, y_j)\Delta A$$

If the number of sections goes to infinity (and each ΔA goes to 0), then the error in this approximation goes to 0 and the double sum approaches a double integral.

$$M_{\overline{x}} = \iint_R (x - \overline{x})f(x, y)\, dA$$

We can get a similar expression if we consider the total moment with respect to a horizontal line at $y = \overline{y}$



$$M_{\overline{y}} = \iint_R (y - \overline{y}) f(x,y)\, dA$$

At the center of mass $(\overline{x},\ \overline{y})$, the total moment should be 0 in all directions:

$$0 = \iint_R (x - \overline{x}) f(x,y)\, dA = \iint_R x f(x,y)\, dA - \overline{x} \iint_R f(x,y)\, dA$$

$$0 = \iint_R (y - \overline{y}) f(x,y)\, dA = \iint_R y f(x,y)\, dA - \overline{y} \iint_R f(x,y)\, dA$$

We can now solve for both $\overline{x}$ and $\overline{y}$

$$\overline{x} = \frac{\iint_R x f(x,y)\, dA}{\iint_R f(x,y)\, dA} \qquad \overline{y} = \frac{\iint_R y f(x,y)\, dA}{\iint_R f(x,y)\, dA}$$

Notice that the denominator $\iint_R f(x,y)\, dA$ equals the mass of region R. If we use the abbreviation δ in place of $f(x,y)$, then the formulas for the coordinates of the centroid become:

$$\overline{x} = \frac{\iint_R x\delta\, dA}{\iint_R \delta\, dA} \qquad \overline{y} = \frac{\iint_R y\delta\, dA}{\iint_R \delta\, dA}$$

Centroids

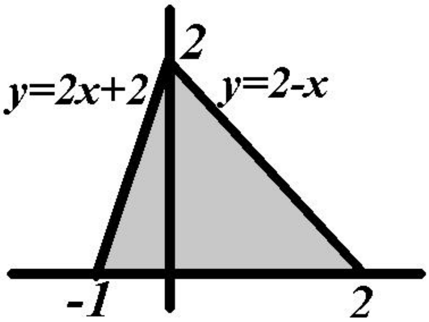
If mass is uniformly distributed throughout R then δ is a constant and the point $(\overline{x},\ \overline{y})$ is called the *centroid* of the region. If δ is a constant then it may be factored out of the integral:

$$\overline{x} = \frac{\iint_R x\delta\, dA}{\iint_R \delta\, dA} = \frac{\delta \iint_R x\, dA}{\delta \iint_R dA} = \frac{1}{\text{Area}(R)} \iint_R x\, dA$$

$$\overline{y} = \frac{\iint_R y\delta\, dA}{\iint_R \delta\, dA} = \frac{\delta \iint_R y\, dA}{\delta \iint_R dA} = \frac{1}{\text{Area}(R)} \iint_R y\, dA$$

Centroid Example

Find the coordinates of the centroid of the triangle with vertices $(-1,\ 0)$, $(0,\ 2)$ and $(2,\ 0)$.

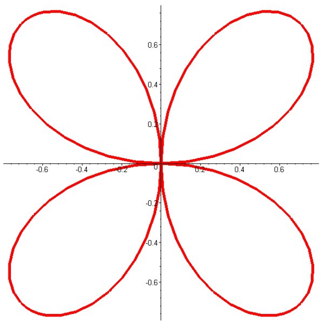


The area of this region is $\frac{1}{2}$(base)(height) $= 3$. Therefore, the coordinates of the centroid are:

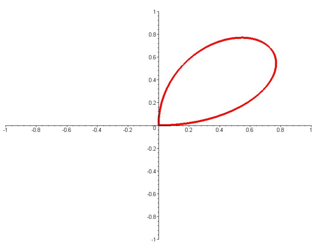
$$\overline{x} = \frac{1}{3}\int_0^2 \int_{(y-2)/2}^{2-y} x\,dx\,dy = \frac{1}{3} \quad \text{and} \quad \overline{y} = \frac{1}{3}\int_0^2 \int_{(y-2)/2}^{2-y} y\,dx\,dy = \frac{2}{3}$$

Centroid Example with Polar Coordinates

The polar coordinate equation $r = \sin(2\theta)$ for $0 \le \theta \le 2\pi$ describes a curve that is sometimes called a *4-leaf rose.*



If we restrict θ to the interval $0 \le \theta \le \frac{\pi}{2}$, we get only one leaf of the rose. Let's call this region R.



Let's calculate the coordinates of the centroid. We start with the area.

$$\text{Area}(R) = \iint_R dA = \int_0^{\pi/2} \int_0^{\sin 2\theta} r\,dr\,d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta\,d\theta = \frac{\pi}{8}$$

By symmetry, we expect the centroid to lie along the line $y = x$, so we only need to calculate $\overline{x}$ since $\overline{y}$ will be the same thing.

$$\begin{aligned}\overline{x} &= \frac{8}{\pi} \iint_R x\,dA = \frac{8}{\pi} \int_0^{\pi/2} \int_0^{\sin 2\theta} r\cos\theta \cdot r\,dr\,d\theta \\ &= \frac{8}{3\pi} \int_0^{\pi/2} \sin^3 2\theta \cos\theta\,d\theta = \frac{64}{3\pi} \int_0^{pi/2} \sin^3\theta \cos^4\theta\,d\theta = \frac{128}{105\pi}\end{aligned}$$

And similarly, $\overline{y} = \frac{128}{105\pi}$