Curvature of the Ellipse

$$\vec{\mathbf{r}} = (\alpha \cos t)\vec{\mathbf{i}} + (\beta \sin t)\vec{\mathbf{j}}$$

$$\vec{\mathbf{v}} = (-\alpha \sin t)\vec{\mathbf{i}} + (\beta \cos t)\vec{\mathbf{j}} \qquad \vec{\mathbf{a}} = (-\alpha \cos t)\vec{\mathbf{i}} + (-\beta \sin t)\vec{\mathbf{j}}$$

$$\kappa = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{v^3} = \frac{\alpha\beta}{\left(\alpha^2 \sin^2 t + \beta^2 \cos^2 t\right)^{3/2}}$$

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If $\alpha = \beta$ then κ is constant.

Suppose $\alpha > \beta$. When is the curvature a maximum?

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Let $f(t) = \alpha^2 \sin^2 t + \beta^2 \cos^2 t$

κ will be maximized when $f(t) = \alpha^2 \sin^2 t + \beta^2 \cos^2 t$ is minimized.

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$$f(t) = \alpha^2 \sin^2 t + \beta^2 \cos^2 t = \beta^2 + (\alpha^2 - \beta^2) \sin^2 t$$

Since $0 \leq \sin^2 t \leq 1$, it follows that:

$$\beta^2 \leq f(t) \leq \alpha^2$$

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The minimum $f(t) = \beta^2$ occurs when $t = n\pi$, so maximum curvature is $\kappa_= \frac{\alpha\beta}{(\beta^2)^{3/2}} = \frac{\alpha}{\beta^2}$.

Circle of Curvature

The radius of curvature was defined as

$$R = \frac{1}{\kappa}$$

For the ellipse, κ is maximized when $t = n\pi$. So, for example, at $t = 0$ and $t = \pi$

$$\vec{\mathbf{r}} = \langle \alpha \cos t,\ \beta \sin t \rangle$$

Let's take $\alpha = 2$ and $\beta = 1$.

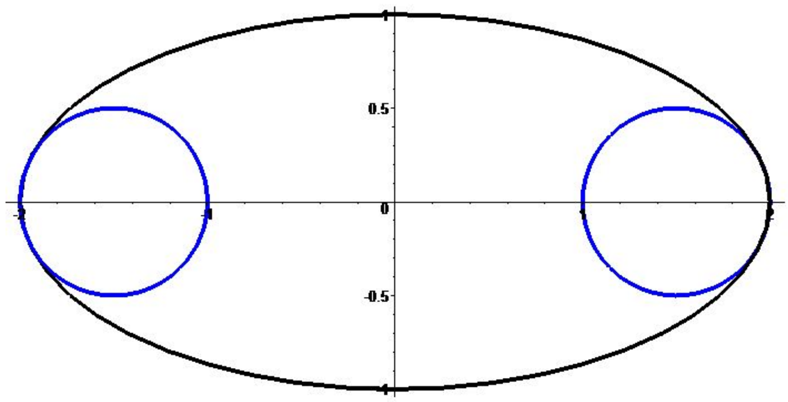
κ is maximized at $t = 0$ and $t = \pi$, so R is minimized at these points.

R is minimized at $(2,\ 0)$ and $(-2,\ 0)$

$$\kappa = \frac{\alpha}{\beta^2} = 2$$

$$R = \frac{1}{\kappa} = \frac{1}{2}$$

Circles of Curvature



R is maximized at $\frac{\pi}{2}$ which occurs at $(0,\ 1)$.