Derivative of a Vector Valued Function

$$\frac{d\vec{\mathbf{f}}}{dt} = \lim_{h \to 0} \frac{\vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)}{h}$$

$$
\begin{aligned}
\vec{\mathbf{f}}(t) &= \langle x(t),\, y(t)\rangle \\
&= x(t)\vec{\mathbf{i}} + y(t)\vec{\mathbf{j}}
\end{aligned}
$$

$$\vec{\mathbf{f}}(t+h) = x(t+h)\vec{\mathbf{i}} + y(t+h)\vec{\mathbf{j}}$$
$$\vec{\mathbf{f}}(t) = x(t)\vec{\mathbf{i}} + y(t)\vec{\mathbf{j}}$$

$$\vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)$$
$$= (x(t+h) - x(t))\vec{\mathbf{i}} + (y(t+h) - y(t))\vec{\mathbf{j}}$$

$$\begin{aligned}
&\frac{\vec{\mathbf{f}}(t+h)-\vec{\mathbf{f}}(t)}{h} \\
&= \frac{x(t+h)-x(t)}{h}\vec{\mathbf{i}} + \frac{y(t+h)-y(t)}{h}\vec{\mathbf{j}}
\end{aligned}$$

Take the limit as $h \to 0$

$$\begin{aligned}
\frac{d\vec{\mathbf{f}}}{dt} &= \frac{dx}{dt}\vec{\mathbf{i}} + \frac{dy}{dt}\vec{\mathbf{j}} \\
&= \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle
\end{aligned}$$

Find the derivative of:

$$\vec{\mathbf{f}}(t) = \langle t^2, t^3 \rangle$$

Find the derivative of:

$$\vec{\mathbf{f}}(t) = \langle t^2,\, t^3 \rangle$$

$$\frac{d\vec{\mathbf{f}}}{dt} = \langle 2t,\, 3t^2 \rangle$$

Find the derivative of:

$$\vec{\mathbf{P}} = \left(e^{2t}\right)\vec{\mathbf{i}} + (\ln t)\vec{\mathbf{j}}$$

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$$\vec{\mathbf{P}} = \left(e^{2t}\right)\vec{\mathbf{i}} + (\ln t)\vec{\mathbf{j}}$$

$$\frac{d\vec{\mathbf{P}}}{dt} = \left(2e^{2t}\right)\vec{\mathbf{i}} + \left(\frac{1}{t}\right)\vec{\mathbf{j}}$$

Find the derivative of:

$$\vec{\mathbf{R}}(t) = \langle \cosh t,\, \sinh t,\, \tanh t \rangle$$

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$$\vec{\mathbf{R}}(t) = \langle \cosh t,\, \sinh t,\, \tanh t \rangle$$

$$\vec{\mathbf{R}}'(t) = \langle \sinh t,\, \cosh t,\, \mathrm{sech}^2 t \rangle$$

Derivative of a Sum of Functions

$$(u+v)' = u' + v'$$

Equivalently,

$$\frac{d}{dt}\big(u(t)+v(t)\big) = \frac{du}{dt} + \frac{dv}{dt}$$

Is this also true if $\vec{\mathbf{u}}(t)$ and $\vec{\mathbf{v}}(t)$ are vectors?

$$\text{Let} \quad \vec{\mathbf{u}}(t) = \langle u_1(t),\, u_2(t)\rangle$$
$$\vec{\mathbf{v}}(t) = \langle v_1(t),\, v_2(t)\rangle$$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \Big\langle u_1(t) + v_1(t),\, u_2(t) + v_2(t)\Big\rangle$$

$$\begin{aligned}
\frac{d}{dt}\big(\vec{\mathbf{u}} + \vec{\mathbf{v}}\big) &= \left\langle \frac{du_1}{dt} + \frac{dv_1}{dt},\, \frac{du_2}{dt} + \frac{dv_2}{dt} \right\rangle \\
&= \left\langle \frac{du_1}{dt},\, \frac{du_2}{dt} \right\rangle + \left\langle \frac{dv_1}{dt},\, \frac{dv_2}{dt} \right\rangle \\
&= \frac{d\vec{\mathbf{u}}}{dt} + \frac{d\vec{\mathbf{v}}}{dt}
\end{aligned}$$

Derivative of a Product of Functions

$$(uv)' = uv' + vu'$$

Equivalently,

$$\frac{d}{dt}\big(u(t)\,v(t)\big) = u\frac{dv}{dt} + v\frac{du}{dt}$$

Is this also true if $\vec{\mathbf{u}}(t)$ and $\vec{\mathbf{v}}(t)$ are vectors?

$$\text{Let} \quad \vec{\mathbf{u}}(t) = \langle u_1(t),\, u_2(t)\rangle$$
$$\vec{\mathbf{v}}(t) = \langle v_1(t),\, v_2(t)\rangle$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2$$

$$\frac{d}{dt}\big(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}\big) = \frac{d}{dt}\left(u_1 v_1 + u_2 v_2\right)$$
$$= \frac{d}{dt}\left(u_1 v_1\right) + \frac{d}{dt}\left(u_2 v_2\right)$$
$$= u_1 \frac{dv_1}{dt} + v_1 \frac{du_1}{dt} + u_2 \frac{dv_2}{dt} + v_2 \frac{du_2}{dt}$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2$$

$$\begin{aligned}
\frac{d}{dt}\big(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}\big) &= \frac{d}{dt}\left(u_1 v_1 + u_2 v_2\right) \\
&= \frac{d}{dt}\left(u_1 v_1\right) + \frac{d}{dt}\left(u_2 v_2\right) \\
&= u_1 \frac{dv_1}{dt} + v_1 \frac{du_1}{dt} + u_2 \frac{dv_2}{dt} + v_2 \frac{du_2}{dt} \\
&= u_1 \frac{dv_1}{dt} + u_2 \frac{dv_2}{dt} + v_1 \frac{du_1}{dt} + v_2 \frac{du_2}{dt}
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\big(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}\big) &= \frac{d}{dt}\left(u_1 v_1 + u_2 v_2\right) \\
&= \frac{d}{dt}\left(u_1 v_1\right) + \frac{d}{dt}\left(u_2 v_2\right) \\
&= u_1 \frac{dv_1}{dt} + u_2 \frac{dv_2}{dt} + v_1 \frac{du_1}{dt} + v_2 \frac{du_2}{dt} \\
&= \langle u_1,\, u_2 \rangle \cdot \left\langle \frac{dv_1}{dt},\, \frac{dv_2}{dt} \right\rangle \\
&\qquad\qquad + \langle v_1,\, v_2 \rangle \cdot \left\langle \frac{du_1}{dt},\, \frac{du_2}{dt} \right\rangle \\
&= \vec{\mathbf{u}} \cdot \frac{d\vec{\mathbf{v}}}{dt} + \vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{u}}}{dt}
\end{aligned}
$$