Arc length Parametrization

The equation $\vec{\mathbf{r}} = \langle \cos t,\ \sin t,\ t\rangle$ describes helical motion around the z-axis. The distance traveled along this path between time$=0$ and time$=\pi$ would be given by:

$$s = \int_0^\pi |\vec{\mathbf{v}}|\, dt = \int_0^\pi |\langle -\sin t,\ \cos t,\ t\rangle\rangle|\, dt = \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\pi$$

If we looked at the distance traveled between time$=0$ and time$=t$, we would obtain:

$$s = \sqrt{2}t$$

We could solve for t in terms of s

$$t = \frac{s}{\sqrt{2}}$$

and then express $\vec{\mathbf{r}}$ in terms of s:

$$\vec{\mathbf{r}} = \left\langle \cos\frac{t}{\sqrt{2}},\ \sin\frac{t}{\sqrt{2}},\ \frac{t}{\sqrt{2}} \right\rangle$$

Instead of expressing position in terms of time, we have expressed it in terms of how far along the curve we are. This is called an *arc length parametrization*. Take a look at the derivative.

$$\frac{d\vec{\mathbf{r}}}{ds} = \left\langle -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$$

This is a tangent vector because $\frac{d\vec{\mathbf{r}}}{ds} = \frac{1}{\sqrt{2}}\langle -\sin t,\ \cos t,\ t\rangle = \frac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}}$. Furthermore, this is a unit tangent vector. We can see this more generally from the Chain Rule:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d\vec{\mathbf{r}}}{ds}\frac{ds}{dt}$$

If we let $v = |\vec{\mathbf{v}}|$ then, $v\frac{d\vec{\mathbf{r}}}{ds} = \vec{\mathbf{v}}$ and $\frac{d\vec{\mathbf{r}}}{ds} = \frac{\vec{\mathbf{v}}}{v} = \vec{\mathbf{T}}$. This is no easier than getting the unit tangent vector directly from $\vec{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{v}$. However, the vector $\frac{d\vec{\mathbf{T}}}{ds}$ will always be perpendicular to the curve. We know this because $\vec{\mathbf{T}} \bullet \vec{\mathbf{T}} = 1$ so $\frac{d}{dt}(\vec{\mathbf{T}} \bullet \vec{\mathbf{T}}) = 0$. The Product Rule then implies that $\vec{\mathbf{T}} \bullet \frac{d\vec{\mathbf{T}}}{ds} = 0$. In the case of the helical path,

$$\frac{d\vec{\mathbf{T}}}{ds} = \left\langle -\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}},\ 0 \right\rangle = -\frac{1}{2}\langle \cos t\ \ \sin t,\ 0\rangle$$

and now you can verify by dot product that this is perpendicular to the unit tangent vector $\vec{\mathbf{T}} = \frac{1}{\sqrt{2}}\langle -\sin t,\ \cos t,\ t\rangle$. $\frac{d\vec{\mathbf{T}}}{ds}$ points in the direction of $\frac{d\vec{\mathbf{T}}}{dt}$ because the Chain Rule says:

$$\frac{d\vec{\mathbf{T}}}{dt} = \frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt} = v\frac{d\vec{\mathbf{T}}}{ds}$$

So, both $\frac{d\vec{\mathbf{T}}}{ds}$ and $\frac{d\vec{\mathbf{T}}}{dt}$ and perpendicular to the tangent line. What is the significance of the length of these vectors? In both cases, we are looking at the rate at which the tangent vector $\vec{\mathbf{T}}$ changes. However, $\vec{\mathbf{T}}$ always has the same length, so any change is due to a change in direction. The faster the tangent vector $\vec{\mathbf{T}}$ changes direction, the bigger its derivative. Consider the following example:

$$\vec{\mathbf{r}} = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct\rangle$$

This is the vector equation of a straight line.

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \angle a,\ b,\ c\rangle \qquad v = |\vec{\mathbf{v}}| = \sqrt{a^2 + b^2 + c^2} \qquad \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{v} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

Since a, b and c are constant, it follows that both $\frac{d\vec{\mathbf{T}}}{ds}$ and $\frac{d\vec{\mathbf{T}}}{dt}$ are zero. This makes sense because for a straight line, $\vec{\mathbf{T}}$ never changes it's length or its direction as we move along the graph.

Curvature

Now let's consider cases where the graph actually curves around. Let's restrict the discussion to two dimensions for the moment. If ϕ is the angle that $\vec{\mathbf{T}}$ makes with $\vec{\mathbf{i}}$, the magnitude of the derivative $\frac{d\phi}{ds}$ is called the *curvature.*

$$\kappa = \left|\frac{d\phi}{ds}\right|$$

This is related to $\frac{d\vec{\mathbf{T}}}{ds}$ by the Chain Rule. If $\vec{\mathbf{T}} = \langle \cos\phi,\ \sin\phi\rangle$, then

$$\frac{d\vec{\mathbf{T}}}{ds} = \frac{d\vec{\mathbf{T}}}{d\phi}\frac{d\phi}{ds} = \langle -\sin\phi,\ \cos\phi\rangle\frac{d\phi}{ds} = \vec{\mathbf{N}}\frac{d\phi}{ds}$$

since $\vec{\mathbf{N}} = \langle -\sin\phi,\ \cos\phi\rangle$ is a unit vector perpendicular to $\vec{\mathbf{T}}$.

$$\left|\frac{d\vec{\mathbf{T}}}{ds}\right| = |\langle -\sin\phi,\ \cos\phi\rangle|\left|\frac{d\phi}{ds}\right| = \kappa|\vec{\mathbf{N}}| = \kappa$$

So, the curvature is the length of $\frac{d\vec{\mathbf{T}}}{ds}$. Since we already know that $\frac{d\vec{\mathbf{T}}}{dt} = v\frac{d\vec{\mathbf{T}}}{ds}$ it follows that $|\vec{\mathbf{T}}'(t)| = v\kappa$, so

$$\kappa = \frac{|T'(t)|}{v}$$

The vector $\frac{d\vec{\mathbf{T}}}{ds}$ will be called the *curvature vector.* It is perpendicular to the tangent to the curve and it's length is the curvature.

Let's do an example where we make use of the formula $\kappa = \frac{|\vec{\mathbf{T}}'(t)|}{v}$.

Curvature of a Circle

The following vector equation describes a circle of radius α in the xy plane centered arund the origin:

$$\vec{\mathbf{r}} = \langle \alpha \cos t,\ \alpha \sin t \rangle$$

The velocity vector, speed and unit tangent vectors are given by;

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \langle -\alpha \sin t,\ \alpha \cos t \rangle \qquad v = |\vec{\mathbf{v}}| = \alpha \qquad \vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{v}}}{v} = \langle -\sin t,\ \cos t \rangle$$

The curvature is therefore:

$$\kappa = \frac{|\vec{\mathbf{T}}'(t)|}{v} = \frac{|\langle -\cos t,\ -\sin t \rangle|}{v} = \frac{1}{\alpha}$$

So, the bigger the radius, the smaller the curvature. We can also rewrite this relationship as $\alpha = \frac{1}{\kappa}$. For a circle, this quantity is constant. For other curves, the quatity $\frac{1}{\kappa}$ is called the *radius of curvature* and may vary from point to point.

Tangential and Normal Components of Acceleration

For the circle, the acceleration vector $\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}$ is perpendicular to the the tangent. For curves that are not necessarilty circles, $\vec{\mathbf{a}}$ has both a tangential and a normal component. The normal component is related to the curvature. To see this, start with $\vec{\mathbf{v}} = v\vec{\mathbf{T}}$ and then take the derivative to get the accelation.

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}(v\vec{\mathbf{T}}(t)) = v'\vec{\mathbf{T}} + v\vec{\mathbf{T}}'(t)$$

We already know that $\frac{d\vec{\mathbf{T}}}{ds} = v\kappa\vec{\mathbf{N}}$ so,

$$\vec{\mathbf{a}} = v'\vec{\mathbf{T}} + v^2\kappa\vec{\mathbf{N}}$$

Thus, v' is the tangential component of acceleration and $v^2\kappa$ is the normal component of acceleration. We can use this to get an alternative formula for the curvature.

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = \vec{\mathbf{v}} \times (v'\vec{\mathbf{T}} + v^2\kappa\vec{\mathbf{N}}) = v'\vec{\mathbf{v}} \times \vec{\mathbf{T}} + v^2\kappa\vec{\mathbf{v}} \times \vec{\mathbf{N}}$$

The vector $\vec{\mathbf{v}} \times \vec{\mathbf{T}} = v\vec{\mathbf{T}} \times \vec{\mathbf{T}}$ is the zero vector, so we are left with:

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = v^2\kappa\vec{\mathbf{v}} \times \vec{\mathbf{N}} = v^2\kappa(v\vec{\mathbf{T}} \times \vec{\mathbf{N}}) = v^3\kappa\vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

$$|\vec{\mathbf{v}} \times \vec{\mathbf{a}}| = v^3\kappa|\vec{\mathbf{T}} \times \vec{\mathbf{N}}| = v^3\kappa$$

This gives us the following formula for curvature:

$$\kappa = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{v^3}$$

We have already calculated the curvature of a circle, but let's see how this formula would work in that case. Since $\vec{\mathbf{r}} = (\alpha\cos t)\vec{\mathbf{i}} + (\alpha\sin t)\vec{\mathbf{j}}$ it follows that

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = (-\alpha\sin t)\vec{\mathbf{i}} + (\alpha\cos t)\vec{\mathbf{j}} \qquad \vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = (-\alpha\cos t)\vec{\mathbf{i}} + (-\alpha\sin t)\vec{\mathbf{j}} \qquad \vec{\mathbf{v}}\times\vec{\mathbf{a}} = \alpha^2\vec{\mathbf{k}}$$

$$\kappa = \frac{|\vec{\mathbf{v}}\times\vec{\mathbf{a}}|}{v^3} = \frac{|\alpha^2\vec{\mathbf{k}}|}{\alpha^3} = \frac{1}{\alpha}$$

This agrees with the formula we obtained for the curvature when we used $\kappa = \frac{\vec{\mathbf{T}}'(t)}{v}$ and is approximately the same amount of work. However, the formula $\kappa = \frac{|\vec{\mathbf{v}}\times\vec{\mathbf{a}}|}{v^3}$ is much more practical for other curves. Let's try calculating the curvature for the ellipse.

Curvature of the Ellipse

The following equation describes an ellipse in the xy plane, centered around the origin.

$$\vec{\mathbf{r}} = (\alpha\cos t)\vec{\mathbf{i}} + (\beta\sin t)\vec{\mathbf{j}}$$

Let's calculate the curvature:

$$\vec{\mathbf{v}} = (-\alpha\sin t)\vec{\mathbf{i}} + (\beta\cos t)\vec{\mathbf{j}} \qquad \vec{\mathbf{a}} = (-\alpha\cos t)\vec{\mathbf{i}} + (-\beta\sin t)\vec{\mathbf{j}} \qquad \vec{\mathbf{v}}\times\vec{\mathbf{a}} = \alpha\beta\vec{\mathbf{k}}$$

$$\kappa = \frac{|\vec{\mathbf{v}}\times\vec{\mathbf{a}}|}{v^3} = \frac{\alpha\beta}{\left(\alpha^2\sin^2 t + \beta^2\cos^2 t\right)^{3/2}}$$

If $\alpha = \beta$ then κ is constant. However this is the case of a circle and we have already dealt with that case. So, let's take $\alpha \neq \beta$, say $\alpha > \beta$. Now the curvature varies from point to point. when is the curvature a maximum?

κ will be maximized when $f(t) = \alpha^2\sin^2 t + \beta^2\cos^2 t$ is minimized. Start by rewriting $f(t) = \beta^2 + (\alpha^2 - \beta^2)\sin^2 t$. Since $0 \le \sin^2 t \le 1$, it follows that:

$$\beta^2 \le f(t) \le \alpha^2$$

The minimum $f(t) = \beta^2$ occurs when $t = n\pi$, so maximum curvature is $\kappa = \frac{\alpha\beta}{(\beta^2)^{3/2}} = \frac{\alpha}{\beta^2}$.

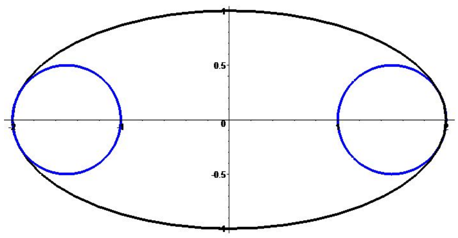
The maximum $f(t) = \alpha^2$ occurs when $t = \frac{2n+1}{2}\pi$, so minimum curvature is $\kappa = \frac{\alpha\beta}{(\alpha^2)^{3/2}} = \frac{\beta}{\alpha^2}$.

Circle of Curvature

The radius of curvature was defined as $R = \frac{1}{\kappa}$. For the circle, this was a constant. What does this mean for other curves? Let's go back to the ellipse. The maximum curvature was $\kappa = \frac{\alpha}{\beta^2}$ so this means the the minimum radius of curvature is $R = \frac{\beta^2}{\alpha}$. Let's look at a specific example. Suppose $\alpha = 2$ and $\beta = 1$. This ellipse can also be described by the equation $\frac{x^2}{4} + \frac{y^2}{1} = 1$. The vertices $(2,\ 0)$ and $(-2,\ 0)$ occur at $t = 0$ and $t = \pi$ respectively, so we get a minimum radius of curvature $R = \frac{1}{2}$ at these points. The minimum curvature was $\kappa = \frac{\beta}{\alpha^2}$ so the maximum radius of curvature will be $R = \frac{\alpha^2}{\beta} = \frac{2^2}{1} = 4$. The vertices

$(0,\ 1)$ and $(0,\ -1)$ at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ will get a maximum radius of curvature of $R = 4$ at these points.

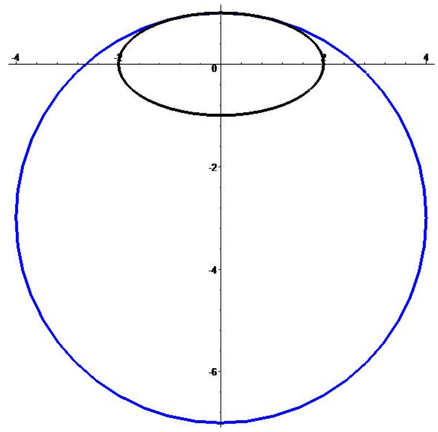
We define the *circle of curvature* at a point on a curve in the following way. If R is the radius of curvature at that point, go a distance of R on the concave side of the normal line through that point (perpendicular to the tangent line) and draw a circle of radius R at the point. So, the circles of curvature at the vertices $(2,\ 0)$ and $(-2,\ 0)$ would be $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{1}{4}$ and $\left(x + \frac{3}{2}\right)^2 + y^2 = \frac{1}{4}$. Take a look at the graphs:



We can see that when we draw the circle of curvature at a point, we get a circle that is tangent to the curve. Let's try this at a different point. At $t = \frac{\pi}{2}$, we are at the vertex $(0,\ 1)$ and the radius of curvature is $R = 4$. So, to draw the circle of curvature, we go down a distance of 4 from this point and draw the circle of radius 4 at this lower point. The equation of this circle of curvature would be:

$$x^2 + (y + 3)^2 = 16$$

Here is the graph:



Again, we see that the circle of curvature at $(0,\ 1)$ is a circle tangent to this point.

Curvature Example - Helix

Earlier, we looked at the helix described by the equation:

$$\vec{\mathbf{r}} = \langle \cos t,\ \sin t,\ t \rangle$$

Let's calculate the curvature using the formula $\kappa = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{v^3}$

$$\vec{\mathbf{v}} = \langle -\sin t,\ \cos t,\ 1\rangle \qquad \vec{\mathbf{a}} = \langle -\cos t,\ -\sin t,\ 0\rangle$$

$$|\vec{\mathbf{v}}| = \sqrt{2} \qquad |\vec{\mathbf{v}} \times \vec{\mathbf{a}}| = |\langle \sin t,\ \cos t,\ -1\rangle| = \sqrt{2}$$

$$\kappa = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

This is a case where we could have used the curvature vector directly. We saw earlier that: $\frac{d\vec{\mathbf{r}}}{ds} = \left\langle -\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$ and the length of this vector is $\frac{1}{2}$