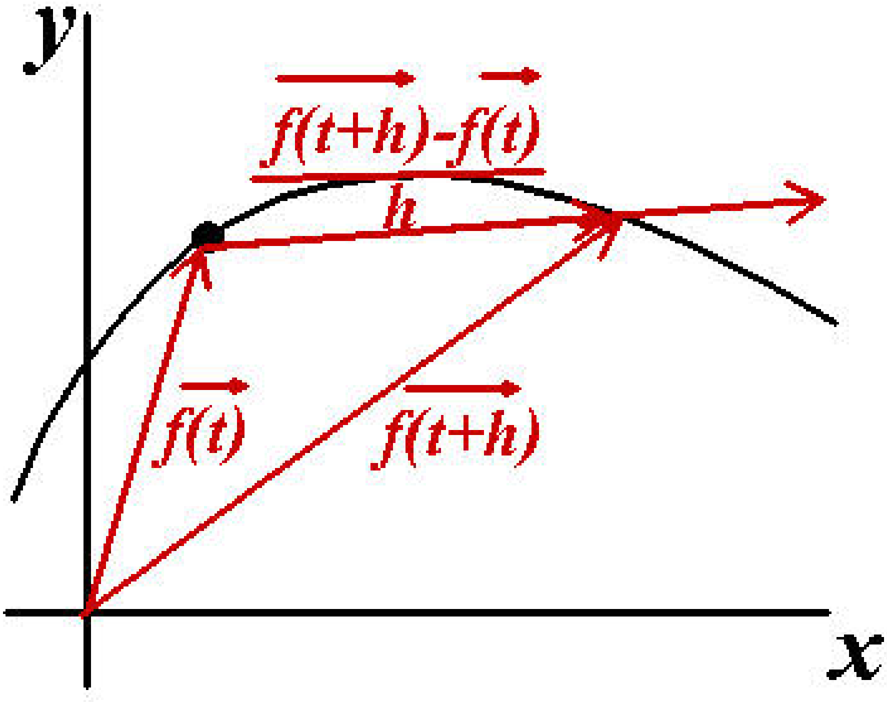
Derivatives of Vector-Valued Functions



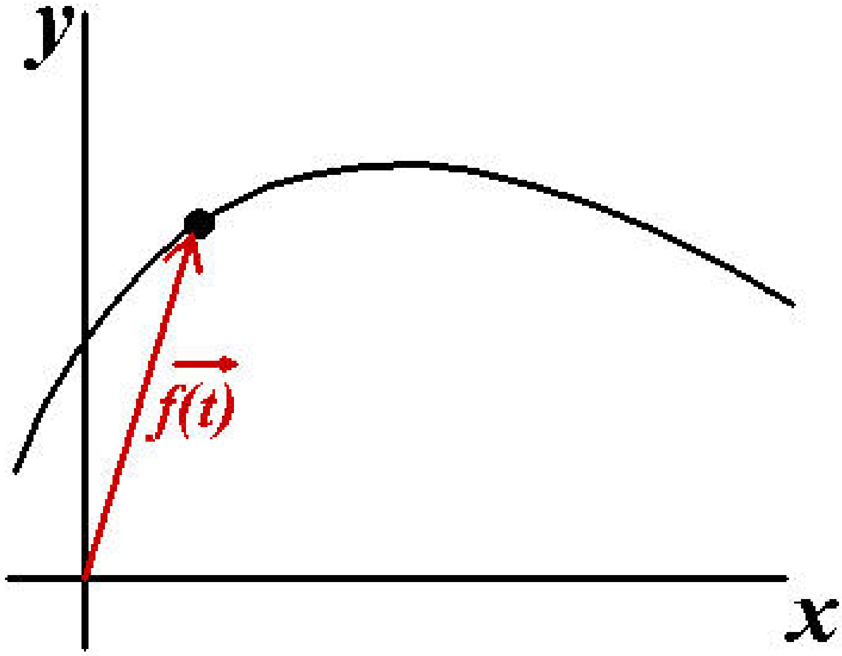
Example: If $\vec{\mathbf{f}}(t) = \langle \sin t,\ \cos 2t \rangle$ then $\vec{\mathbf{f}}'(t) = \langle \cos t,\ -2\sin 2t \rangle$

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}(t) = \langle x(t),\ y(t) \rangle$$

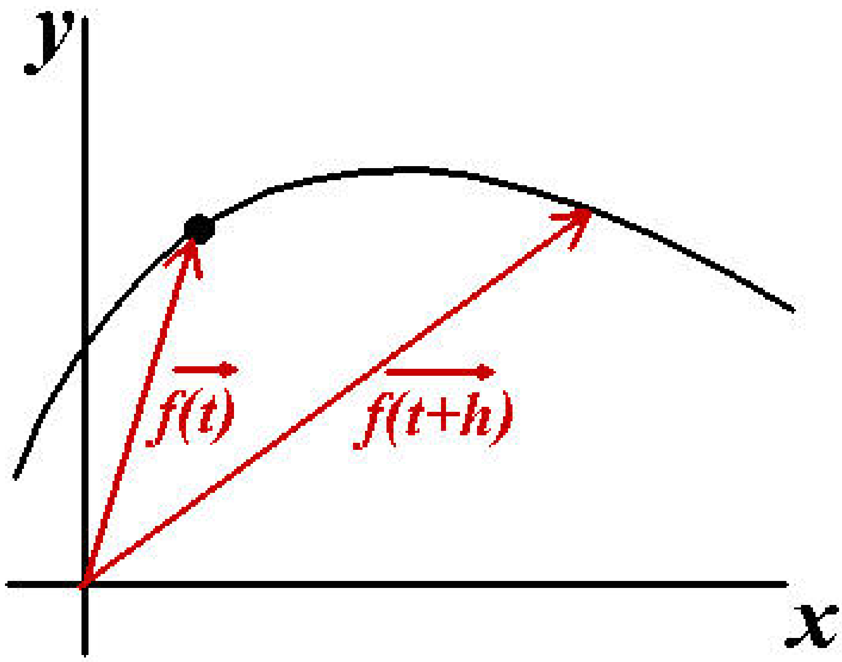
$$\frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{f}}'(t) = \langle x'(t),\ y'(t) \rangle$$

$$\vec{\mathbf{f}}'(\mathbf{t}) = \lim_{h \to 0} \frac{\vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)}{h}$$

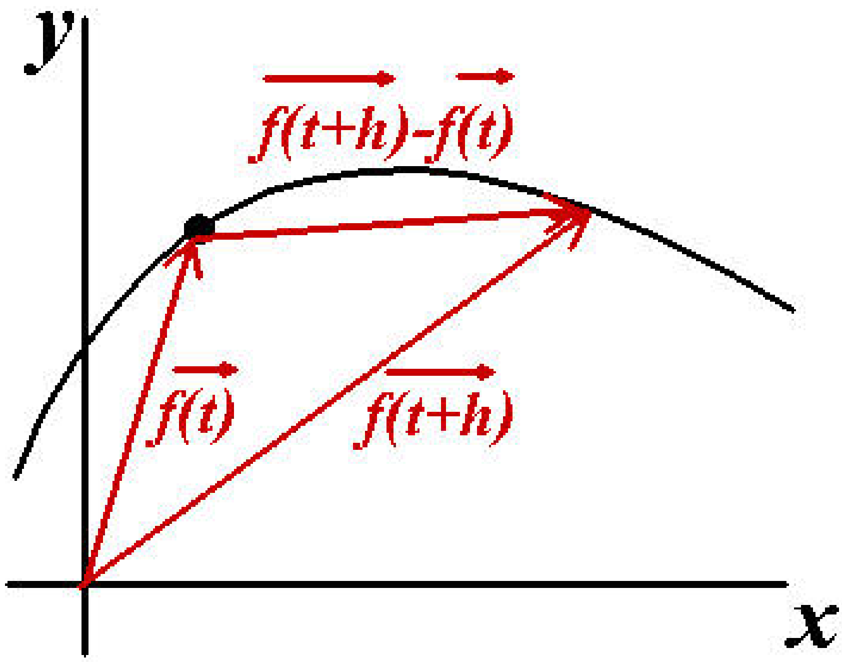
$$\vec{\mathbf{f}}(t) = \langle x(t),\ y(t) \rangle$$



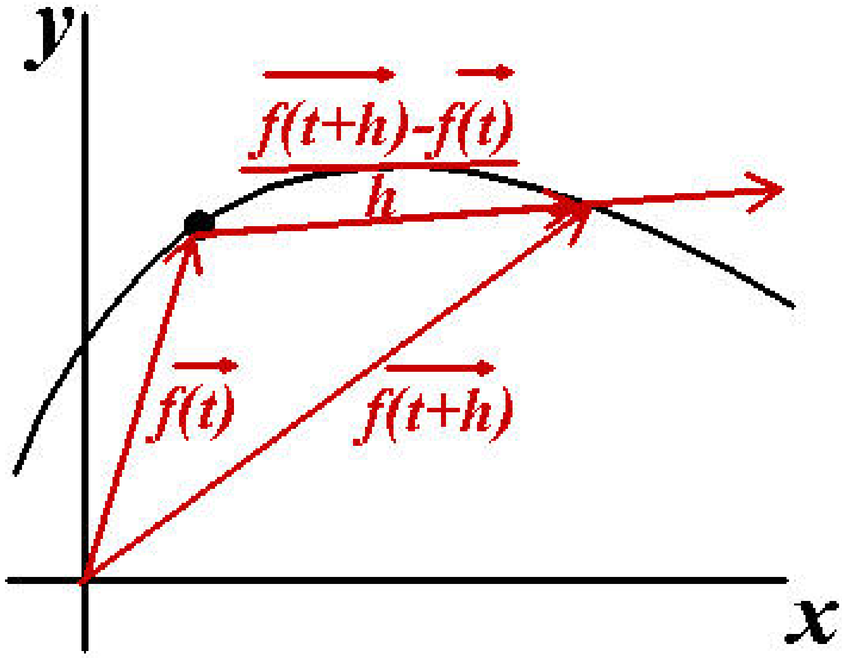
$$\vec{\mathbf{f}}(t+h) = \langle x(t+h),\ y(t+h)\rangle$$

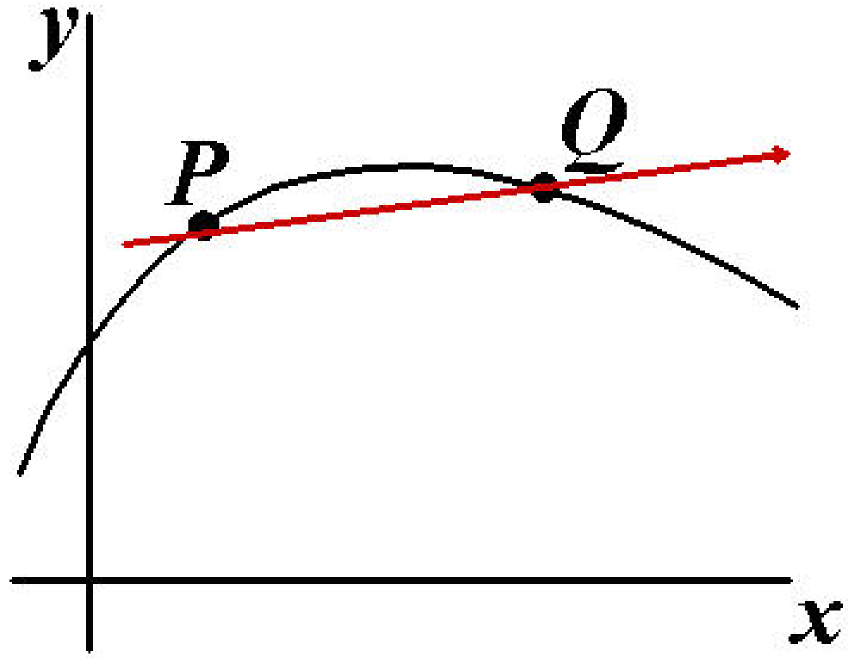


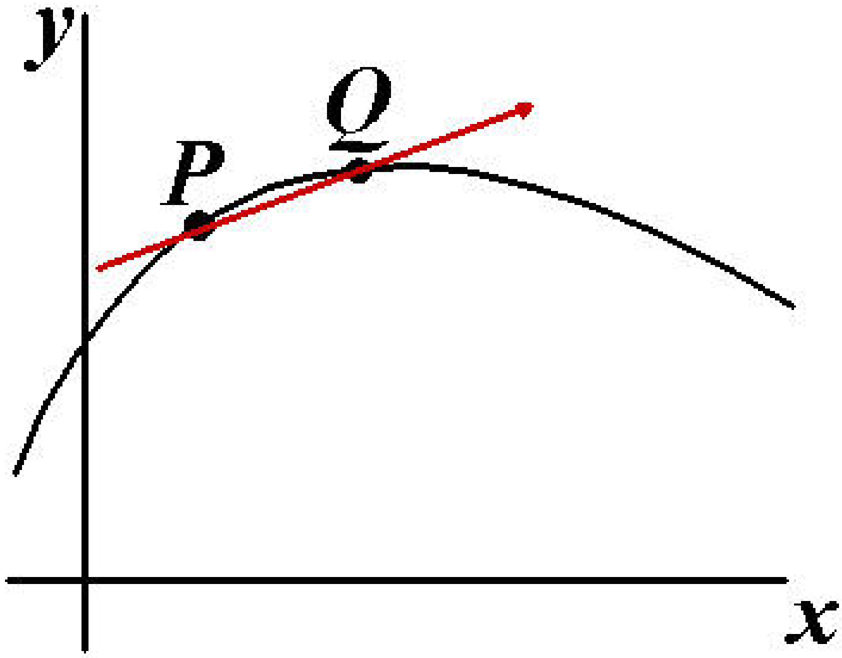
$$\text{Change In Position} = \vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)$$

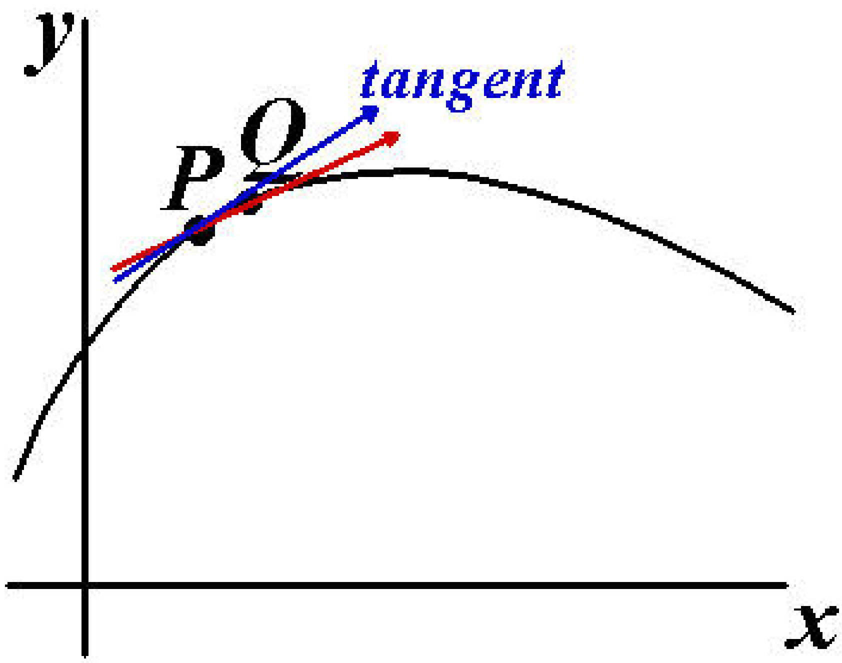


$$\frac{\vec{\mathbf{f}}(t+h)-\vec{\mathbf{f}}(t)}{h} = \frac{1}{h}(\vec{\mathbf{f}}(t+h)-\vec{\mathbf{f}}(t))$$



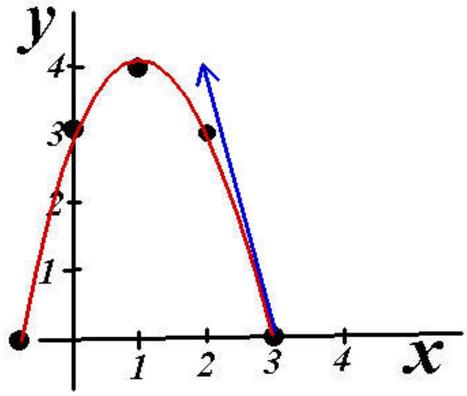

y
P
Q
x


y
Q
P
x


y
tangent
Q
P
x

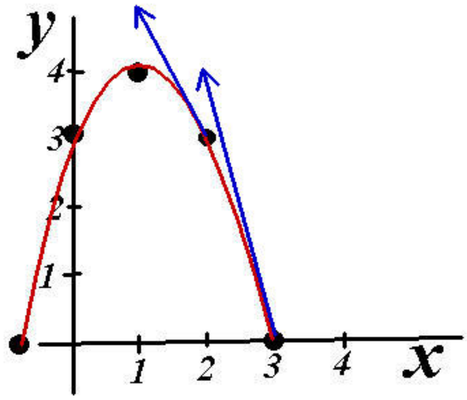
$$\vec{\mathbf{f}}(t) = \langle 3 - t,\ 4t - t^2 \rangle$$

$$\vec{\mathbf{f}}'(t) = \langle -1,\ 4 - 2t \rangle \qquad \vec{\mathbf{f}}'(0) = \langle -1,\ 4 \rangle$$



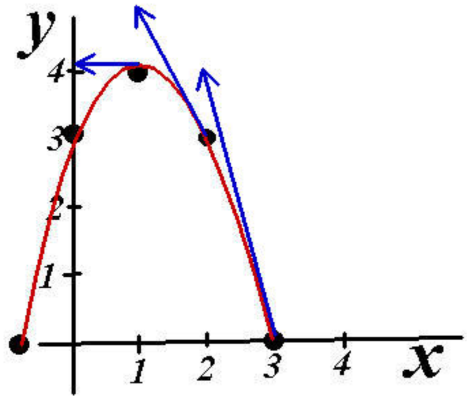
$$\vec{\mathbf{f}}(t) = \langle 3 - t,\ 4t - t^2 \rangle$$

$$\vec{\mathbf{f}}'(t) = \langle -1,\ 4 - 2t \rangle \qquad \vec{\mathbf{f}}'(1) = \langle -1,\ 2 \rangle$$



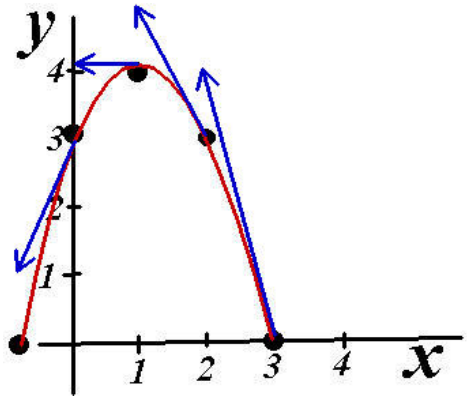
$$\vec{\mathbf{f}}(t) = \langle 3 - t,\ 4t - t^2 \rangle$$

$$\vec{\mathbf{f}}'(t) = \langle -1,\ 4 - 2t \rangle \qquad \vec{\mathbf{f}}'(2) = \langle -1,\ 0 \rangle$$

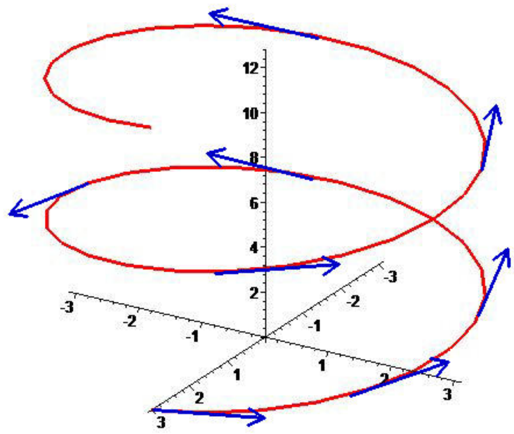


$$\vec{\mathbf{f}}(t) = \langle 3 - t,\ 4t - t^2 \rangle$$

$$\vec{\mathbf{f}}'(t) = \langle -1,\ 4 - 2t \rangle \qquad \vec{\mathbf{f}}'(3) = \langle -1,\ 2 \rangle$$



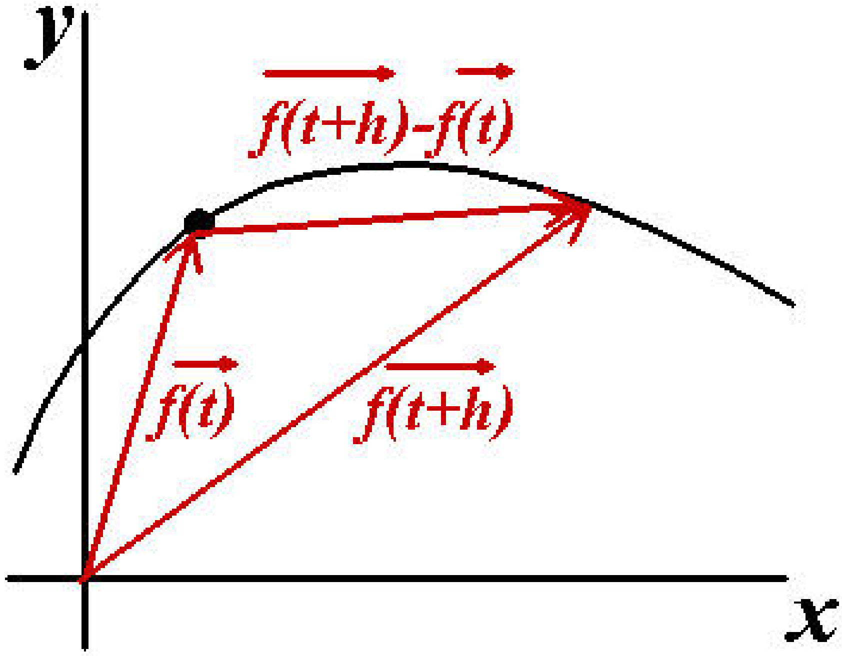
$$\vec{\mathbf{f}}(t) = \langle 3\cos t,\ 3\sin t,\ t\rangle \qquad \vec{\mathbf{f}}'(t) = \langle -3\sin t,\ 3\cos t,\ 1\rangle$$



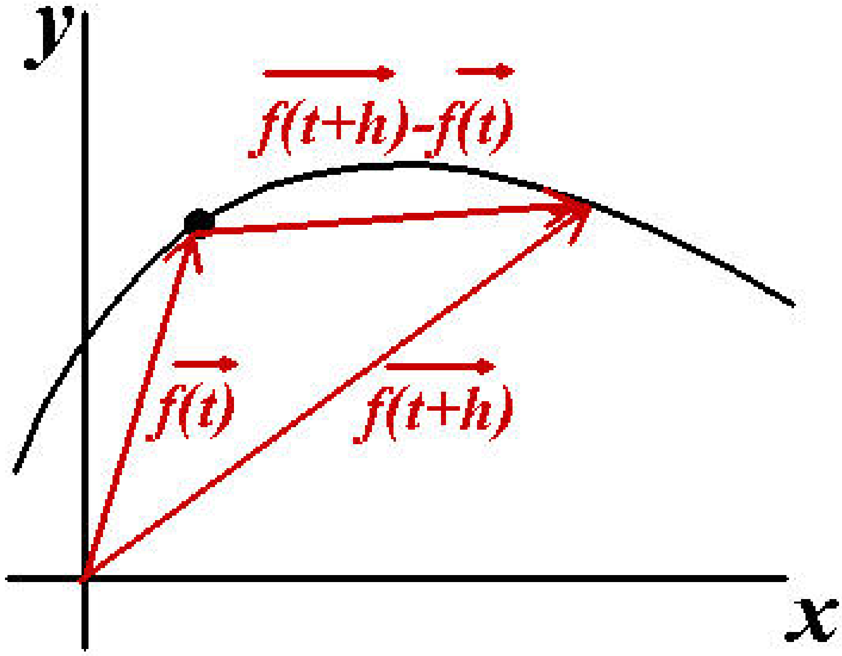
$$\vec{\mathbf{f}}(t) = \langle x(t),\ y(t),\ z(t) \rangle \qquad \vec{\mathbf{f}}'(t) = \langle x'(t),\ y'(t),\ z'(t) \rangle$$

$$|\vec{\mathbf{f}}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

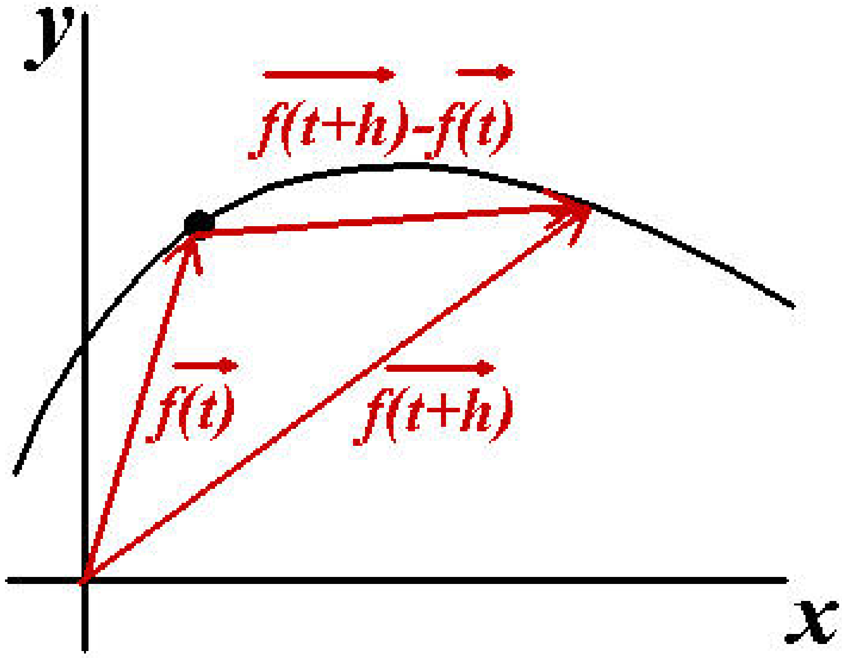
$$\text{Change In Position} = \vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)$$



$$\text{distance} = |\vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)|$$



$$\frac{\text{distance}}{\text{time}} = \frac{|\vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)|}{h}$$



Instantaneous speed:

$$|\vec{\mathbf{f}}'(t)| = \lim_{h \to 0} \frac{|\vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)|}{h}$$

If position as a function of time is:

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}(t)$$

Then the velocity vector is:

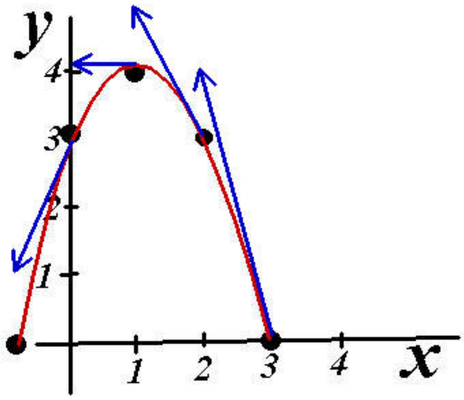
$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}$$

The speed is:

$$v = |\vec{\mathbf{v}}|$$

$$\vec{\mathbf{r}} = \langle 3 - t,\ 4t - t^2 \rangle$$

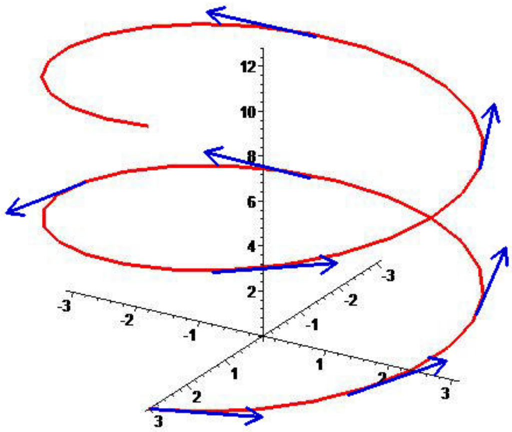
$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \langle -1,\ 4 - 2t \rangle \qquad |\vec{\mathbf{v}}| = \sqrt{1 + (4 - 2t)^2}$$



$$\vec{\mathbf{r}} = \langle 3\cos t,\ 3\sin t,\ t\rangle \qquad \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \langle -3\sin t,\ 3\cos t,\ 1\rangle$$

f

$$|\vec{\mathbf{v}}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 1^2} = \sqrt{10}$$



If position as a function of time is:

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}(t)$$

Then the velocity vector is:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}$$

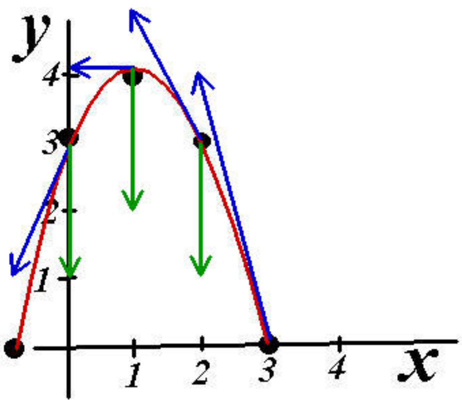
The acceleration vector is the derivative of the velocity vector

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2}$$

$$\vec{\mathbf{r}} = \langle 3 - t,\ 4t - t^2 \rangle$$

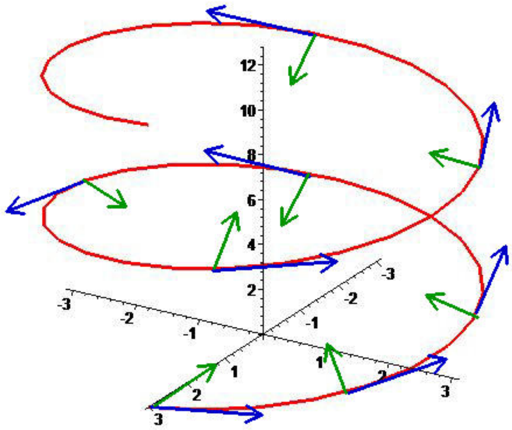
$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \langle -1,\ 4 - 2t \rangle \qquad |\vec{\mathbf{v}}| = \sqrt{1 + (4 - 2t)^2}$$

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \langle 0,\ -2 \rangle$$



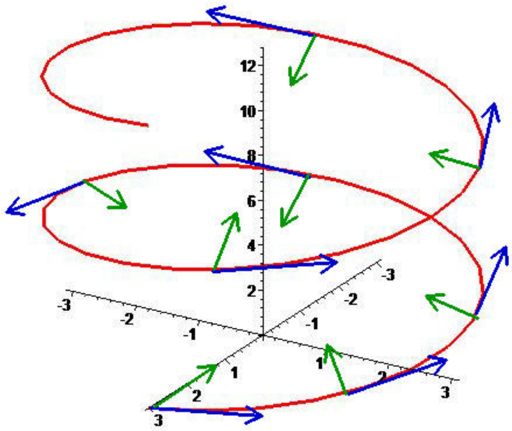
$$\vec{\mathbf{r}} = \langle 3\cos t,\ 3\sin t,\ t\rangle \qquad \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \langle -3\sin t,\ 3\cos t,\ 1\rangle$$

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \langle -3\cos t,\ -3\sin t,\ 0\rangle$$

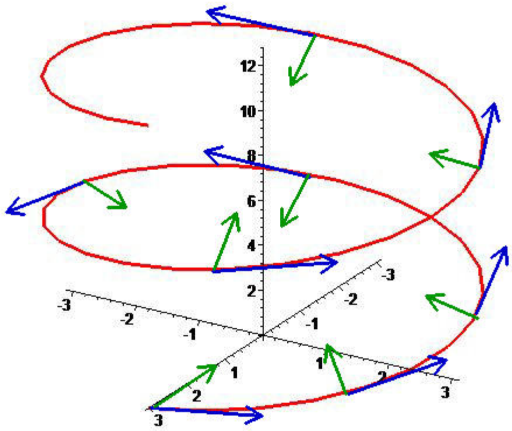


$$\vec{\mathbf{v}} = \langle -3\sin t,\ 3\cos t,\ 1\rangle \qquad \vec{\mathbf{a}} = \langle -3\cos t,\ -3\sin t,\ 0\rangle$$

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{a}} = 9\sin t\cos t - 9\sin t\cos t = 0$$



Theorem: If an object is traveling at constant speed then the acceleration vector is perpendicular to the velocity vector at all points.



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$$|\vec{\mathbf{v}}| = C$$

$$|\vec{\mathbf{v}}|^2 = C^2$$

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$$|\vec{\mathbf{v}}|^2 = C^2$$

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{v}} = C^2$$

Theorem: If an object is traveling at constant speed then the acceleration vector is perpendicular to the velocity vector at all points.

$$|\vec{\mathbf{v}}| = C$$

$$|\vec{\mathbf{v}}|^2 = C^2$$

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{v}} = C^2$$

$$\frac{d}{dt}\left(\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}\right) = \frac{d}{dt}\left(C^2\right)$$

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$$\frac{d}{dt}\left(\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}\right) = \frac{d}{dt}\left(C^2\right)$$

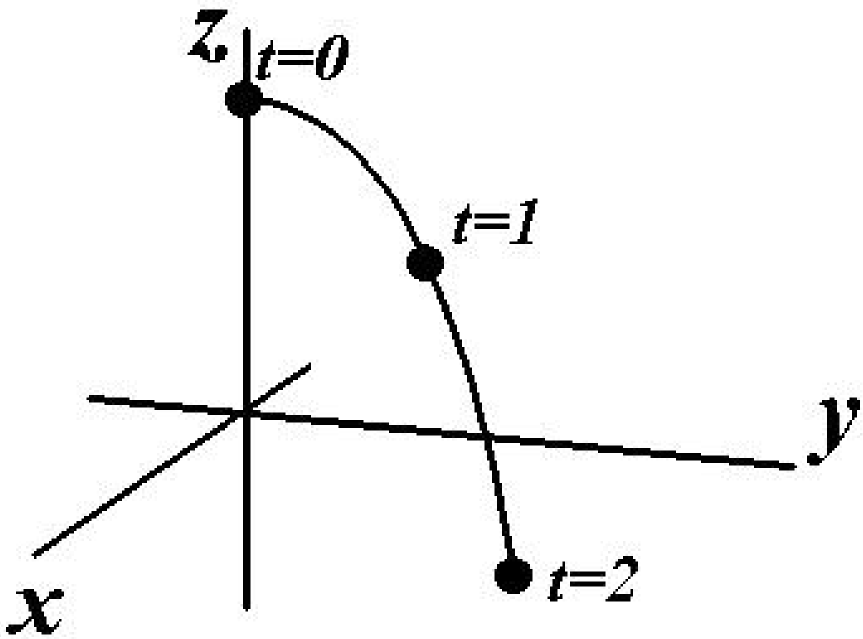
$$\vec{\mathbf{v}} \bullet \frac{d\vec{\mathbf{v}}}{dt} + \vec{\mathbf{v}} \bullet \frac{d\vec{\mathbf{v}}}{dt} = 0$$

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{a}} + \vec{\mathbf{v}} \bullet \vec{\mathbf{a}} = 0$$

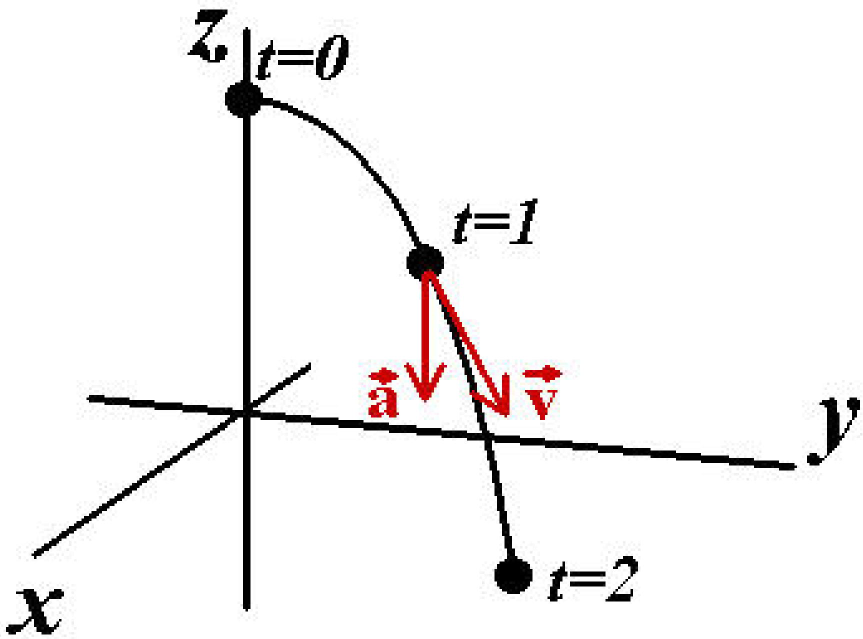
$$2\vec{\mathbf{v}} \bullet \vec{\mathbf{a}} = 0$$

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{a}} = 0$$

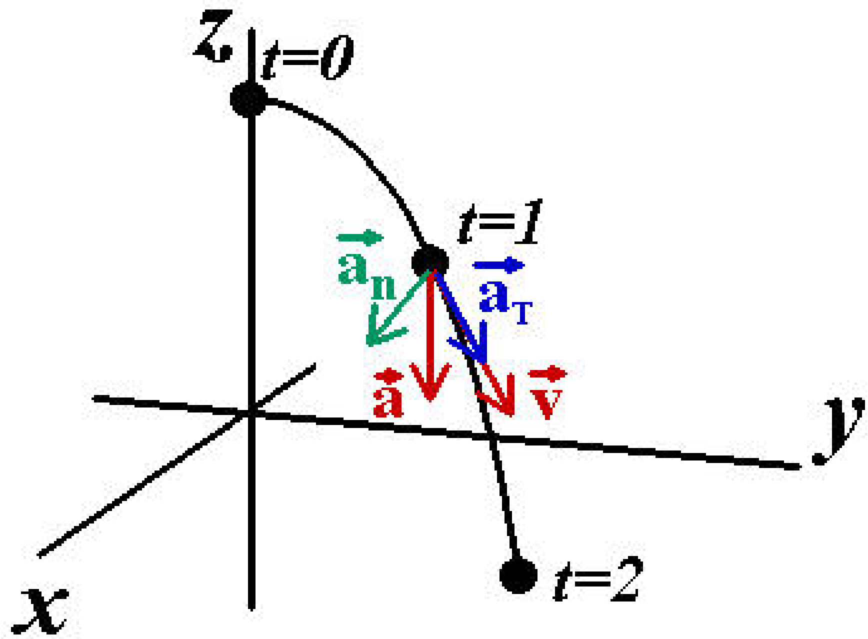
$$\vec{\mathbf{r}} = \langle t,\ t,\ 4 - t^2 \rangle \qquad \vec{\mathbf{v}} = \langle 1,\ 1,\ -2t \rangle \qquad \vec{\mathbf{a}} = \langle 0,\ 0,\ -2 \rangle$$



At $t = 1$, $\vec{\mathbf{v}} = \langle 1,\ 1,\ -2 \rangle$ $\vec{\mathbf{a}} = \langle 0,\ 0,\ -2 \rangle$



$$\vec{\mathbf{v}} = \langle 1,\ 1,\ -2\rangle \qquad \vec{\mathbf{a}} = \langle 0,\ 0,\ -2\rangle \qquad \vec{\mathbf{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_n$$



$$\vec{\mathbf{v}} = \langle 1,\ 1,\ -2\rangle \qquad \vec{\mathbf{a}} = \langle 0,\ 0,\ -2\rangle \qquad \vec{\mathbf{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_n$$

$$\begin{aligned} \vec{\mathbf{a}}_T &= \operatorname{proj}_{\vec{\mathbf{v}}} \vec{\mathbf{a}} \\ &= \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{a}}}{\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}}\, \vec{\mathbf{v}} \end{aligned}$$

$$\vec{\mathbf{v}} = \langle 1,\ 1,\ -2\rangle \qquad \vec{\mathbf{a}} = \langle 0,\ 0,\ -2\rangle \qquad \vec{\mathbf{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_n$$

$$\begin{aligned}
\vec{\mathbf{a}}_T &= \operatorname{proj}_{\vec{\mathbf{v}}} \vec{\mathbf{a}} \\
&= \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{a}}}{\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}}\, \vec{\mathbf{v}} \\
&= \frac{4}{6}\langle 1,\ 1,\ -2\rangle \\
&= \left\langle \frac{2}{3},\ \frac{2}{3},\ -\frac{4}{3} \right\rangle
\end{aligned}$$

$$\vec{\mathbf{v}} = \langle 1,\ 1,\ -2 \rangle \qquad \vec{\mathbf{a}} = \langle 0,\ 0,\ -2 \rangle \qquad \vec{\mathbf{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_n$$

$$\vec{\mathbf{a}}_T = \left\langle \frac{2}{3},\ \frac{2}{3},\ -\frac{4}{3} \right\rangle$$

If $\vec{\mathbf{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_n$ then:

$$\vec{a}_n = \vec{\mathbf{a}} - \vec{\mathbf{a}}_T = \langle 0,\ 0,\ -2 \rangle - \left\langle \frac{2}{3},\ \frac{2}{3},\ -\frac{4}{3} \right\rangle = -\frac{2}{3} \langle 1,\ 1,\ 1 \rangle$$

$$\vec{\mathbf{r}}(t) = \langle \cos t,\ \sin t,\ \sqrt{2}\cos t \rangle$$