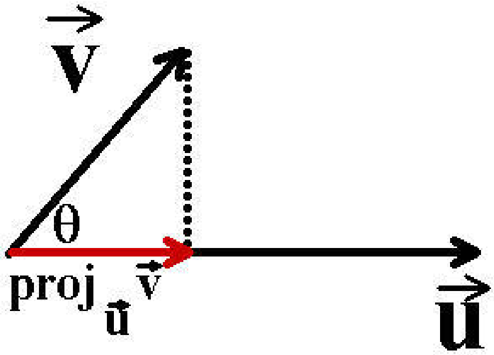
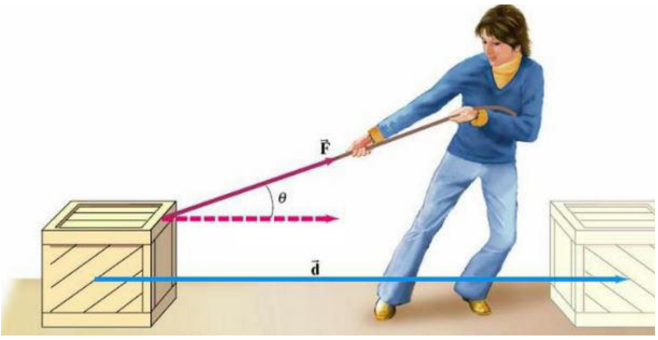
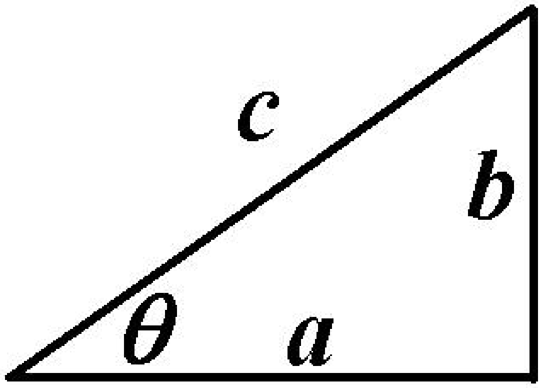
# Projection of one vector on another



**Work = (Force)(Distance)**

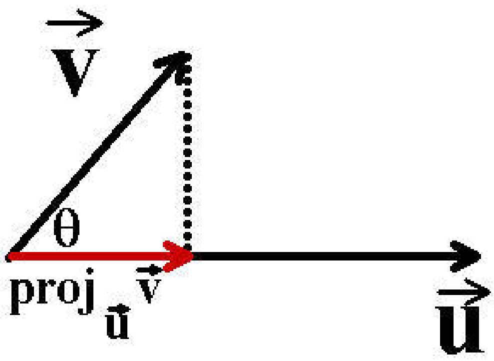


## Use of cosine to find adjacent side




$$\cos\theta = \frac{a}{c} \qquad a = c\cos\theta$$

**Use of cosine to find a projection**




$$|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}| = |\vec{\mathbf{v}}|\cos\theta$$

## Definition of Dot Product

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{u}}|\,|\vec{\mathbf{v}}|\cos\theta$$

## Relationship of Projection to Dot Product

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{u}}|\,|\vec{\mathbf{v}}| \cos\theta$$

$$\frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{v}}| \cos\theta = |\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|$$

Note that $\frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}}$ is a *unit vector* in the direction of $\vec{\mathbf{u}}$.

If $\vec{\mathbf{u}} = \langle u_1,\ u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2,\ u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2,\ v_3 \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2,\ \ldots\ , u_n \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2,\ \ldots\ , v_n \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2,\ \dots,u_n \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2,\ \dots,v_n \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \sum_{k=1}^{n} u_k v_k$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2,\ \ldots\ , u_n\rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2,\ \ldots\ , v_n\rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \sum_{k=1}^{n} u_k v_k$$

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{u}} = \sum_{k=1}^{n} v_k u_k$$

The dot product is commutative:

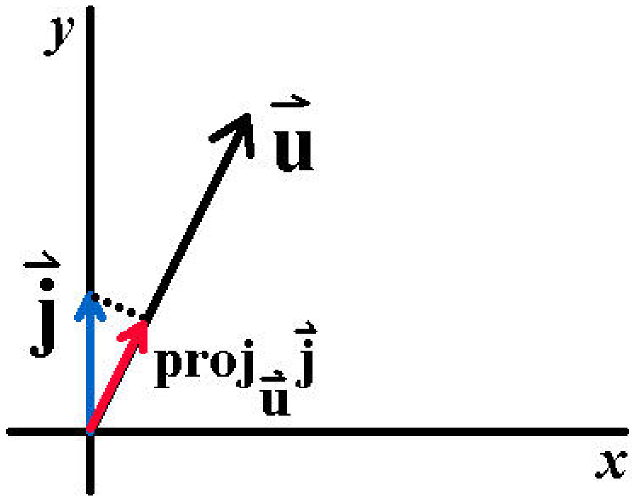
$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \vec{\mathbf{v}} \bullet \vec{\mathbf{u}}$$

$$\text{If } \vec{\mathbf{v}} = \langle v_1,\ v_2,\ v_3 \rangle \quad \text{then} \quad |\vec{\mathbf{v}}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$|\vec{\mathbf{v}}|^2 = v_1^2 + v_2^2 + v_3^2 = \langle v_1,\ v_2,\ v_3 \rangle \bullet \langle v_1,\ v_2,\ v_3 \rangle = \vec{\mathbf{v}} \bullet \vec{\mathbf{v}}$$

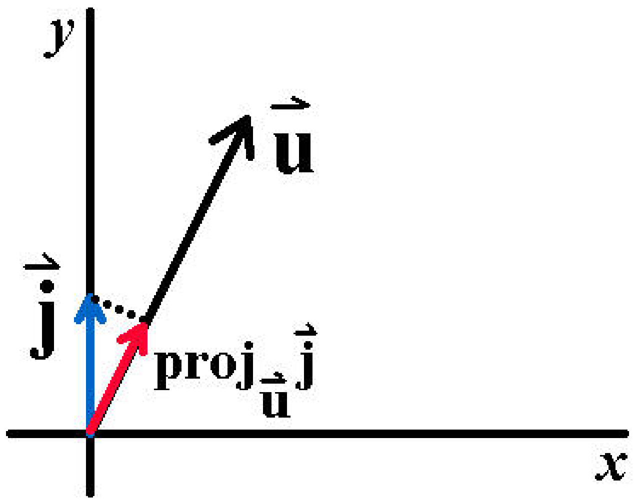
**Example** Let $\vec{\mathbf{u}} = \langle 3,\ 4\rangle$ and let $\vec{\mathbf{j}} = \langle 0,\ 1\rangle$

Calculate the length of the projection of $\vec{\mathbf{j}}$ in the direction of $\vec{\mathbf{u}}$



$$\vec{\mathbf{u}} = \langle 3,\ 4\rangle \qquad \vec{\mathbf{j}} = \langle 0,\ 1\rangle$$

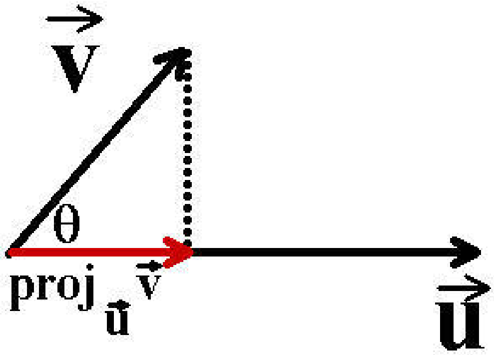
$$|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{j}}| = \vec{\mathbf{j}} \bullet \frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}} = \langle 0,\ 1\rangle \bullet \frac{1}{5}\langle 3,\ 4\rangle = \langle 0,\ 1\rangle \bullet \langle 3/5,\ 4/5\rangle = \frac{4}{5}$$



We can use the dot product to calculate the length of $\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. How do we calculate the vector $\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ itself?

$\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ points in the same direction as $\vec{\mathbf{u}}$.

Therefore, $\dfrac{\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}}{|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|}$ is the same unit vector as $\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$

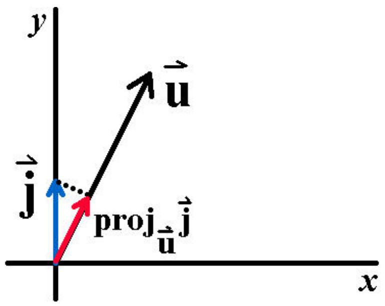


$$\frac{\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}}{|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|} = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$$

$$\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = |\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}| \cdot \frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}} = \vec{\mathbf{v}} \bullet \frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}} \cdot \frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}}{|\vec{\mathbf{u}}|^2}\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}\vec{\mathbf{u}}$$

**Example** Let $\vec{\mathbf{u}} = \langle 3,\ 4 \rangle$ and let $\vec{\mathbf{j}} = \langle 0,\ 1 \rangle$

We have already calculated that the length of the projection of $\vec{\mathbf{j}}$ on $\vec{\mathbf{u}}$ is $\frac{4}{5}$. Now, let's calculate the vector $\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{j}}$ itself.




$$\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{j}} = \frac{\vec{\mathbf{u}} \bullet \vec{\mathbf{j}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}\vec{\mathbf{u}} = \frac{4}{25}\langle 3,\ 4 \rangle$$

**Example**

$$\vec{\mathbf{u}} = \langle 3,\ 4\rangle \qquad \vec{\mathbf{w}} = -\vec{\mathbf{i}} = \langle -1,\ 0\rangle$$

Calculate the dot product

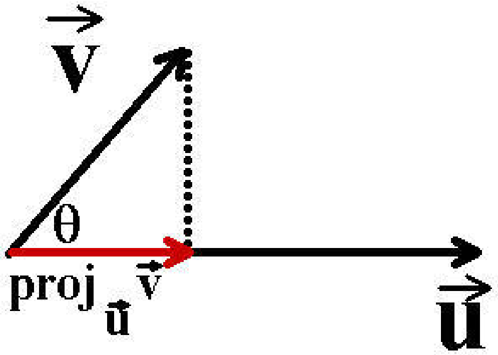
$$\vec{\mathbf{u}} \bullet \vec{\mathbf{w}} = -3$$

What does it mean when the dot product is negative?

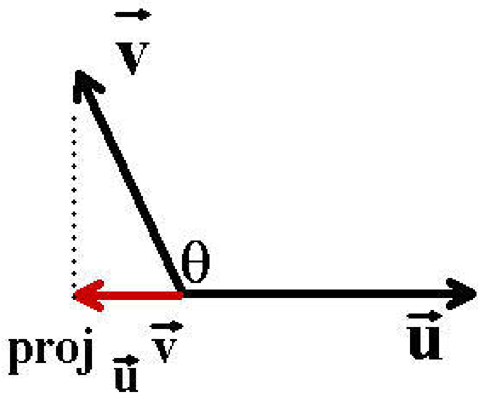
$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

The dot product is only negative when $\cos\theta$ is negative

If $0 < \theta < \frac{\pi}{2}$ then $\cos\theta > 0$ and the dot product is positive



If $\frac{\pi}{2} < \theta < \pi$ then $\cos\theta < 0$ and the dot product is negative



What does it mean if the dot product is zero?

What does it mean if the dot product is zero?

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = 0$$

$$|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = 0$$

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have nonzero length, then $\cos\theta = 0$

$$\theta = \frac{\pi}{2}$$

When two vectors are perpendicular (orthogonal) then the dot product is zero.

**Example**

Let $\vec{\mathbf{x}} = \langle 2,\ 1 \rangle$. Find a vector $\vec{\mathbf{y}}$ that is perpendicular to $\vec{\mathbf{x}}$

$$\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0$$

$$\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0$$

$$\langle 2,\ 1 \rangle \bullet \langle y_1,\ y_2 \rangle = 0$$

$$2y_1 + y_2 = 0$$

$$y_2 = -2y_1$$

$$\vec{\mathbf{y}} = \langle y_1,\ y_2 \rangle = \langle y_1,\ -2y_1 \rangle = y_1 \langle 1,\ -2 \rangle$$

Thus, any scalar multiple of $\langle 1,\ -2 \rangle$ is perpendicular to $\langle 2,\ 1 \rangle$

The vector $\langle x_1,\ x_2\rangle$ is always perpendicular to $\langle x_2,\ -x_1\rangle$

$\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ notation

$$\begin{aligned}\langle x_1,\ x_2\rangle &= \langle x_1,\ 0\rangle + \langle 0,\ x_2\rangle \\ &= x_1\langle 1,\ 0\rangle + x_2\langle 0,\ 1\rangle \\ &= x_1\vec{\mathbf{i}} + x_2\vec{\mathbf{j}}\end{aligned}$$

The vectors $x_1\vec{\mathbf{i}} + x_2\vec{\mathbf{j}}$ and $x_2\vec{\mathbf{i}} - x_1\vec{\mathbf{j}}$ are always perpendicular to each other

Determinant notation:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If you need a vector perpendicular to $x_1\vec{\mathbf{i}} + x_2\vec{\mathbf{j}}$, calculate the determinant:

$$\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} \\ x_1 & x_2 \end{vmatrix} = x_2\vec{\mathbf{i}} - x_1\vec{\mathbf{j}}$$