Tangent Planes and Differentials

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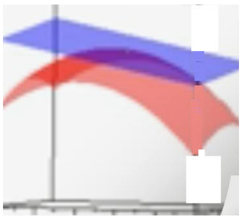
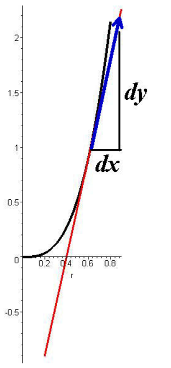
The idea of a tangent line was fundamental to an understanding of first year calculus. In higher dimension, the tangent plane plays an equally important role in the study of multivariate calculus.



Tangent Vectors

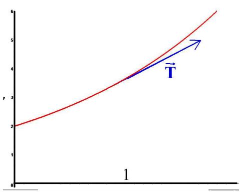
To get the equation of a tangent plane, we first need to discuss tangent vectors. Let's start in two dimensions. If $y = f(x)$ then dy is the change in the height of the tangent line if x is changed by an amount dx.



A vector tangent to the curve would have coordinates $\langle dx,\ dy\rangle$. Since $y' = \frac{dy}{dx}$ we can write the tangent vector as:

$$\langle dx,\ dy\rangle = \langle dx,\ y'\,dx\rangle = \langle 1,\ y'\rangle\,dx = (1\vec{\mathbf{i}} + y'\vec{\mathbf{j}})\,dx$$

Thus, any scalar multiple of the vector $\langle 1,\ y'\rangle$ will be a tangent vector. As a quick example, let's say we wanted a vector $\vec{\mathbf{T}}$ that was tangent to the curve $y = 2e^{x/2}$ at the point $x = 1$.

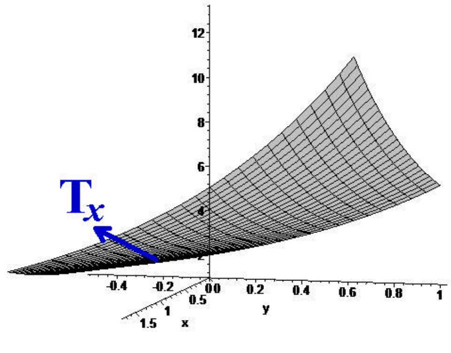


$y' = e^{x/2}$, so at $x = 1$ y' will be $\sqrt{e}$ and a tangent vector would be any scalar multiple of:

$$\vec{\mathbf{T}} = \langle 1,\ \sqrt{e}\rangle$$

Now, let's move on to a three dimensional problem. Suppose $z = 2e^{x/2+y}$. If $y = 0$, the surface intersects the xz-plane as a curve $z = 2e^{x/2}$. This time, the tangent vector will be $\vec{\mathbf{T}}_x = 1\vec{\mathbf{i}} + \frac{\partial z}{\partial x}\vec{\mathbf{k}}$ since the vertical direction is now in the direction of $\vec{\mathbf{k}}$. So, at the point $(1,\ 0,\ 2\sqrt{e})$, the tangent vector is:

$$\vec{\mathbf{T}}_x = 1\vec{\mathbf{i}} + \sqrt{e}\vec{\mathbf{k}} = \langle 1,\ 0,\ \sqrt{e}\rangle$$

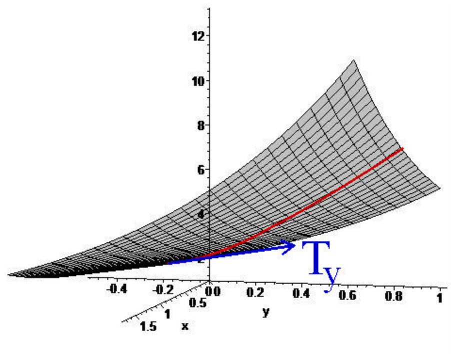


Similarly, if the take $x = 1$ and vary y, the surface $z = 2e^{x/2+y}$ becomes the curve $z = 2e^{1/2+y}$ in a plane parallel to the yz plane. and the tangent vector will be:

$$\vec{\mathbf{T}}_y = 1\vec{\mathbf{j}} + \frac{\partial z}{\partial y}\vec{\mathbf{k}}$$

$\frac{\partial z}{\partial y} = 2e^{x/2+y}$ so at the point $(1,\ 0,\ 2\sqrt{e})$, $\frac{\partial z}{\partial y} = 2\sqrt{e}$ and the tangent vector is:

$$\vec{\mathbf{T}}_y = \langle 1,\ 0,\ 2\sqrt{e}\rangle$$

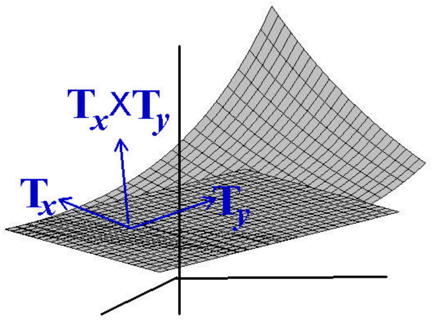


We are ready to get the equation of the plane tangent to this surface at the point. In general, the equation of a planc is given by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where $\langle a,\ b,\ c\rangle$ is a vector perpendicular to the plane.

If $\vec{\mathbf{T}}_x$ and $\vec{\mathbf{T}}_y$ are vectors tangent to the plane then $\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y$ will be perpendicular to the plane.



Let's use this to obtain the plane tangent to $f(x, y) = 2e^{x/2+y}$ at the point $(1,\ 0,\ 2\sqrt{e})$

$$\langle a,\ b,\ c\rangle = \vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & \sqrt{e} \\ 0 & 1 & 2\sqrt{e} \end{vmatrix} = \langle -\sqrt{e},\ -2\sqrt{e},\ 1\rangle$$

So, $a = -\sqrt{e}$, $b = -2\sqrt{e}$ and $c = 1$. Substitute into the equation of the plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\sqrt{e}(x - 1) + 2\sqrt{e}(y - 0) + (1)(z - 2\sqrt{e}) = 0$$

$$z = \sqrt{e}(x + 2y + 1)$$

The height of the tangent plane is sometimes written as $L(x, y)$, which stands for the *linear approximation* of the function.

Let's go to the more general case where $z = f(x, y)$ and we wish to find the equation of the plane tangent to the surface at $(x_0,\ y_0,\ z_0)$ where $z_0 = f(x_0,\ y_0)$. Let's use the abbreviations z_x and z_y for $\frac{\partial z}{\partial x}(x_0,\ y_0)$ respectively. The tangent vectors are now:

$$\vec{\mathbf{T}}_x = \langle 1,\ 0,\ z_x\rangle \quad \text{and} \quad \vec{\mathbf{T}}_y = \langle 0,\ 1,\ z_y\rangle$$

and the vector perpendicular to the tangent plane will be:

$$\langle a,\ b,\ c\rangle = \vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \langle -z_x,\ -z_y,\ 1\rangle$$

Since $a = -z_x$, $b = -z_y$ and $c = 1$ the equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ becomes:

$$-z_x(x - x_0) - z_y(y - y_0) + (1)(z - z_0) = 0$$

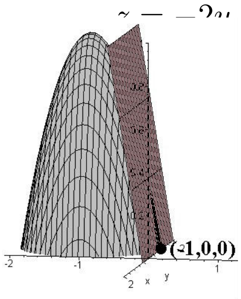
$$z = z_0 + z_x(x - x_0) + z_y(y - y_0)$$

Example

Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x+1)^2 - (y+1)^2$ at the point $(-1,\ 0,\ 0)$.

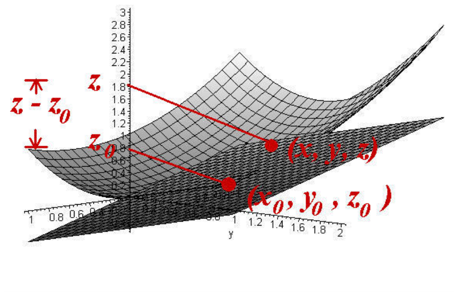
$$z_x = \frac{\partial z}{\partial x}(-1,\ 0) = 0 \qquad\qquad z_y = \frac{\partial z}{\partial y}(-1,\ 0) = -2$$

Substitute into the equation $z = z_0 + z_x(x - x_0) + z_y(y - y_0)$



The Differential

In Calculus I, the differential $dy = y'dx$ was the change in the height of the tangent line. In Calculus III, the differential is the change in the height of the tangent plane. Suppose we move from the point $(x_0,\ y_0,\ z_0)$ to the point $(x,\ y,\ z)$ on the tangent plane. The difference $z - z_0$ is the change in the height of the tangent plane.

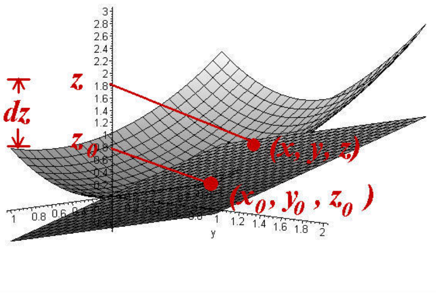


The equation of the tangent plane gives us the formula for $z - z_0$

$$z - z_0 = z_x(x - x_0) + z_y(y - y_0)$$

The change $x - x_0$ is denoted by dx and the change $y - y_0$ is denoted by dy. The differential $z - z_0$ is denoted by dz. This implies that:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$



If dx and dy are small, the change in the height of the tangent plane is a fairly good approximation to the change in the height of a surface. For example, consider the exponential surface we looked at earlier

$$f(x, y) = 2e^{x/2+y}$$

If we change (x, y) from $(1,\ 0)$ to $(1.02,\ 0.04)$, the change in the height of the surface is:

$$f(1.02,\ 0.04) - f(1,\ 0) = 2e^{1.02/2+0.04} - 2e^{1/2+0} = 2e^{0.55} - 2e^{0.5} \approx 0.169$$

Now, let's compare this to the change in the height of the tangent plane. We saw that the plane tangent to this surface at $(1,\ 0)$ is:

$$L(x, y) = \sqrt{e}(x + 2y + 1)$$

The change in the height of the tangent plane will be:

$$L(1.02,\ 0.04) - L(1,\ 0) = \sqrt{e}(1.02 + (2)(0.04) + 1) - \sqrt{e}(1 + 0 + 1) = \sqrt{e}(0.100) \approx 1.65$$

If we use differential notation to do this, then $dx = 1.02 - 1 = 0.02$ and $dy = 0.04 - 0 = 0.04$, so the differential is:

$$dz = \frac{\partial z}{\partial x}(1, 0) \cdot dx + \frac{\partial z}{\partial y}(1, 0) \cdot dy = \sqrt{e} \cdot (0.02) + 2\sqrt{e} \cdot (0.04) = \sqrt{e}(0.100) \approx 1.65$$

If dz approximates the change in the height of the surface, then $\frac{dz}{z}$ approximates the *relative change* in the height of the surface.

Example: According to the Ideal Gas Law, the pressure of a gas is given by:

$$P = \frac{nRT}{V}$$

where n is the number of moles of gas, R is the ideal gas constant, T is the temperature of the gas and V is the volume of the gas. Suppose V is changed from 1,000 liters to 1200 liters and the temperature is changed from 100 degrees to 130 degrees. Let's use the differential to calculate the relative change in pressure, $\frac{dP}{P}$.

$$\frac{\partial P}{\partial T} = \frac{nR}{V} \qquad \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$$

Therefore, the differential dP is given by:

$$dP = \frac{\partial P}{\partial T}dT + \frac{\partial P}{\partial V}dV = \frac{nR}{V}dT - \frac{nRT}{V^2}dV$$

The relative change in pressure is approximated by the differential expression:

$$\frac{dP}{P} = \frac{\frac{nR}{V}dT - \frac{nRT}{V^2}dV}{\frac{nRT}{V}} = \frac{dT}{T} - \frac{dV}{V}$$

Therefore, if we start from $V = 1,000$ and $T = 100$ and we change volume and temperature by the amounts $dV = 200$ and $dT = 30$ then the relative change in pressure is approximately:

$$\frac{dP}{P} = \frac{1}{100}(30) + \frac{200}{1,000} = \frac{1}{10}$$

So a 20 percent increase in volume and a 30 percent increase in temperature results in approximately a 10 percent increase in pressure.