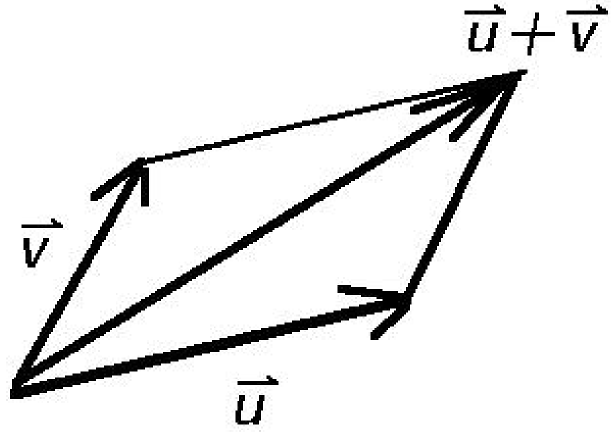
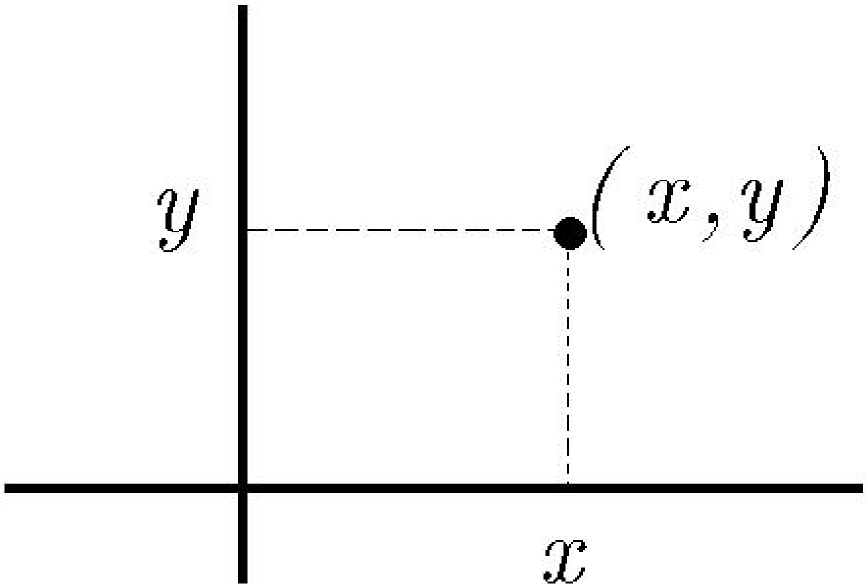
Basic Vector Operations

Dr. Elliott Jacobs

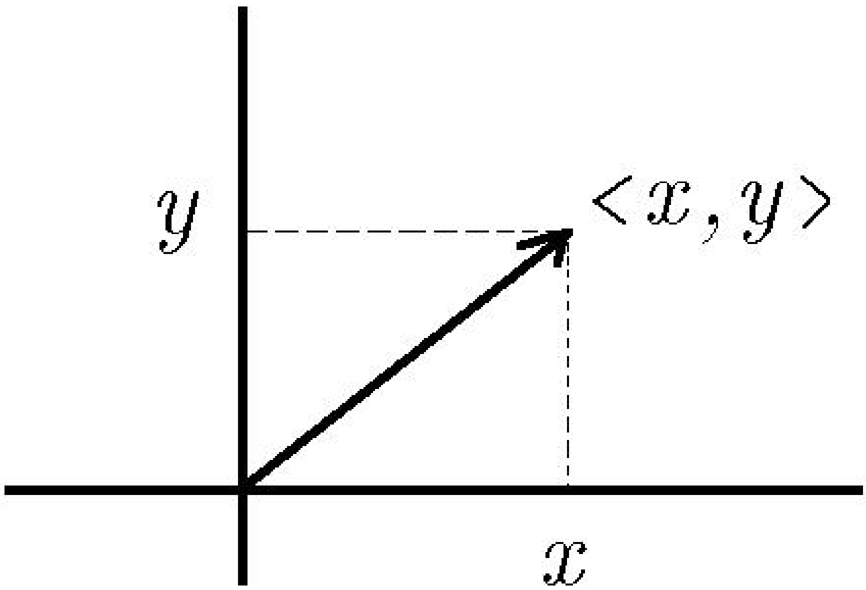


$$(x, y)$$

(x, y) can be represented as a point in the plane

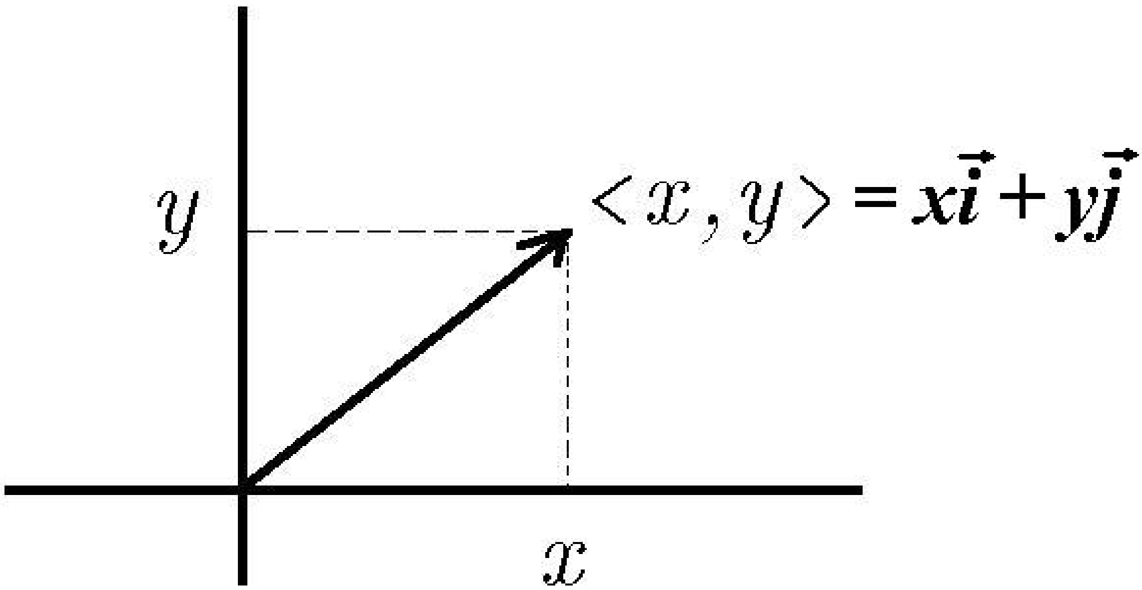


A pair of numbers can be represented by an arrow.



A quantity with both magnitude and direction is called a *vector*

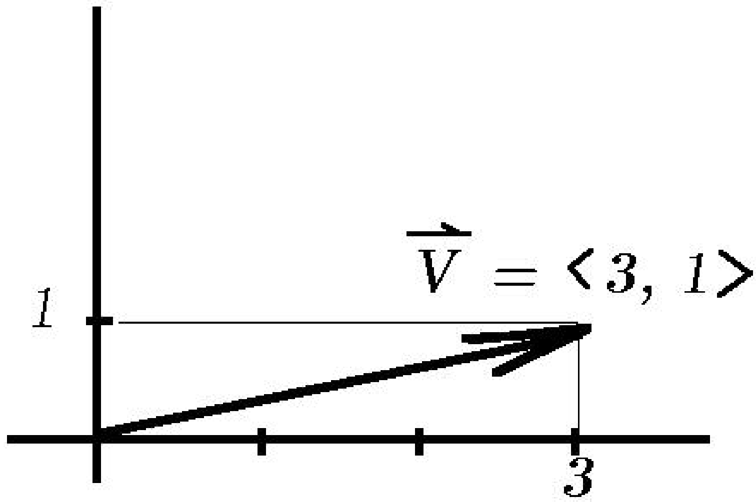
A pair of numbers can be represented by an arrow.



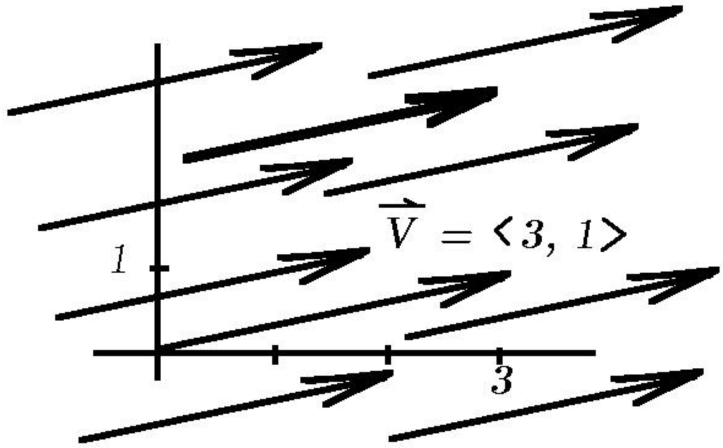
A quantity with both magnitude and direction is called a *vector*

Calculate the length of $\vec{\mathbf{v}}$ with the distance formula

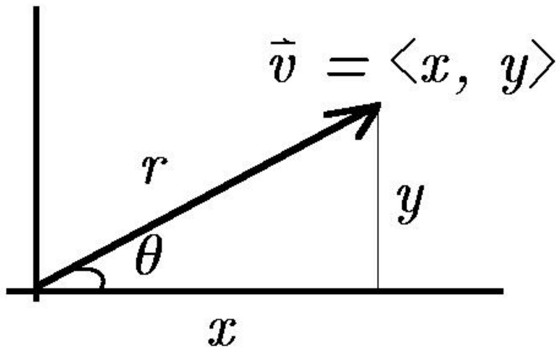
$$|\vec{\mathbf{v}}| = \sqrt{3^2 + 1^2} = \sqrt{10}$$



Any vector parallel to $\vec{\mathbf{v}}$ having the same length is considered to be the same as $\vec{\mathbf{v}}$, even if the tail of the vector does not begin at the origin.

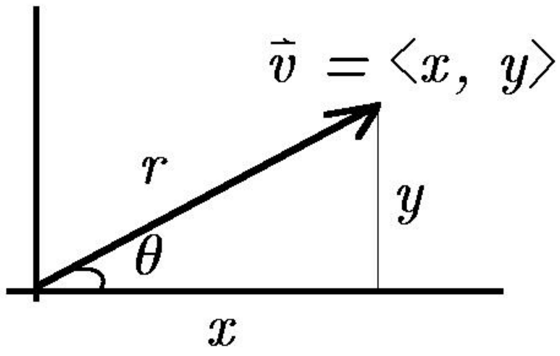


Let θ be the angle that a vector $\vec{\mathbf{v}} = \langle x,\, y\rangle$ makes with the x-axis. $r = |\vec{\mathbf{v}}| = \sqrt{x^2 + y^2}$ is the length.



$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

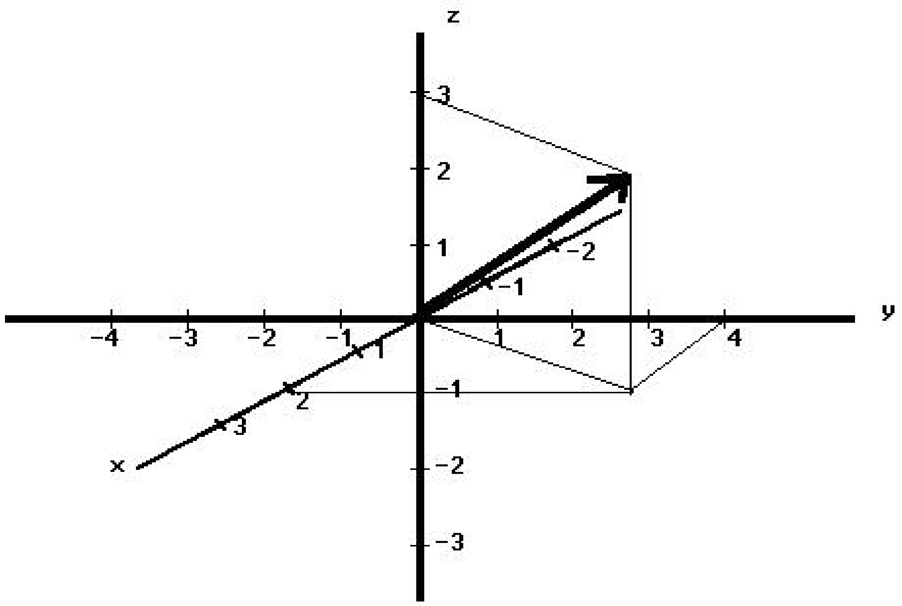
Let θ be the angle that a vector $\vec{\mathbf{v}} = \langle x,\, y \rangle$ makes with the x-axis. $r = |\vec{\mathbf{v}}| = \sqrt{x^2 + y^2}$ is the length.



$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\vec{\mathbf{v}} = \langle x,\, y \rangle = \langle r\cos\theta,\, r\sin\theta \rangle$$

Vectors are easily generalized to higher dimension.



$$\vec{\mathbf{v}} = \langle 2,\ 4,\ 3 \rangle$$

Definition of Scalar Multiplication

Let $\vec{\mathbf{v}} = \langle x, \, y \rangle$.
Let c be a *scalar* (in other words, c is a number).

$$c\vec{\mathbf{v}} = c \, \langle x, \, y \rangle = \langle cx, \, cy \rangle$$

Let $\vec{\mathbf{v}} = \langle x,\, y,\, z \rangle$

$$c\vec{\mathbf{v}} = c\langle x,\, y,\, z \rangle = \langle cx,\, cy,\, cz \rangle$$

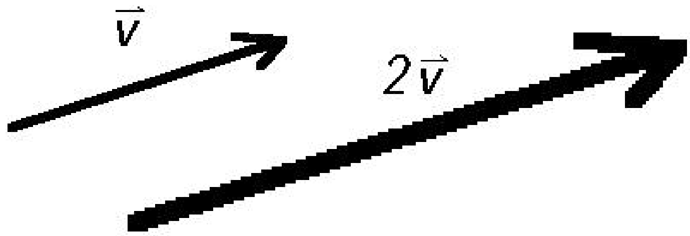
More generally, if $\vec{\mathbf{v}} = \langle v_1, v_2, \ldots, v_n \rangle$ then

$$c\vec{\mathbf{v}} = \langle cv_1, cv_2, \ldots, cv_n \rangle$$

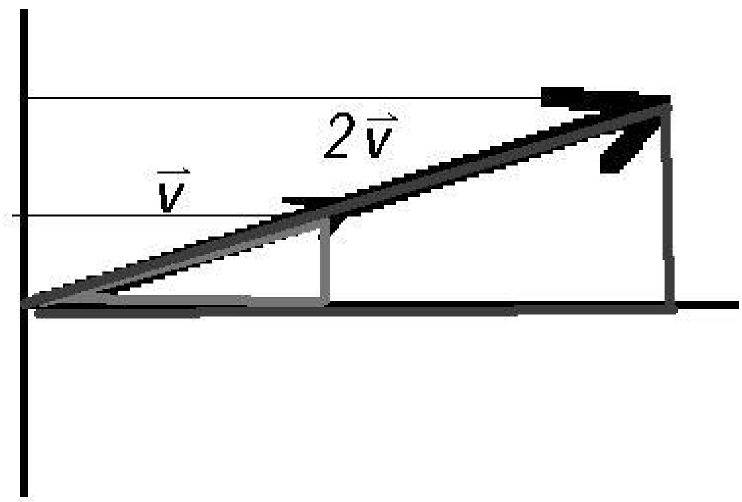
Definition of Scalar Multiplication

Motivation

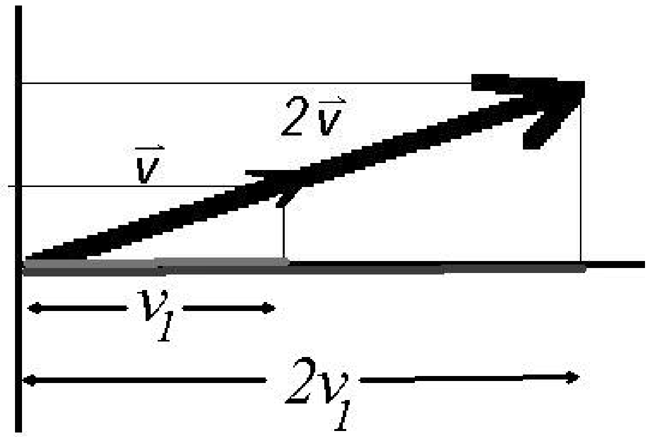
Let $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$. We would like to define the vector $2\vec{\mathbf{v}}$ so that it is in the same direction as $\vec{\mathbf{v}}$ but twice as long.



Drop perpendiculars to form two similar triangles where the hypotenuse of the larger triangle is twice that of the smaller triangle.

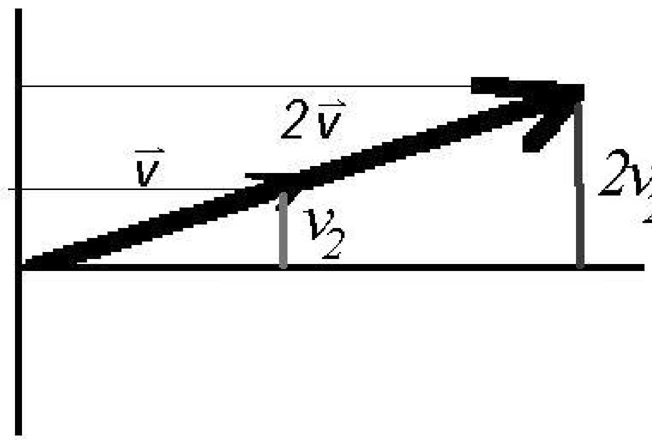


By similar triangles, the first coordinate of $2\vec{\mathbf{v}}$ must be twice the first coordinate of $\vec{\mathbf{v}}$.



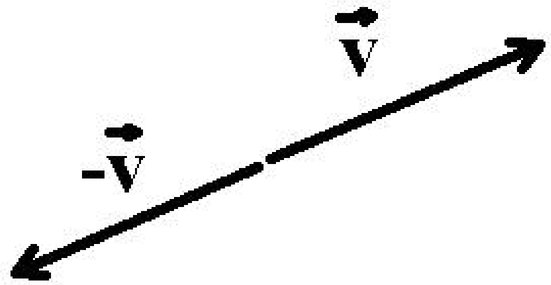
By similar triangles, the second coordinate of $2\vec{\mathbf{v}}$ must be twice the second coordinate of $\vec{\mathbf{v}}$.

$$\vec{\mathbf{v}} = \langle v_1,\, v_2 \rangle \qquad 2\vec{\mathbf{v}} = \langle 2v_1,\, 2v_2 \rangle$$



Multiplication by a negative number reverses direction

$$-\vec{\mathbf{v}} = -\langle v_1,\ v_2 \rangle = \langle -v_1,\ -v_2 \rangle$$



A vector of length $= 1$ is a *unit vector*

If $\vec{\mathbf{v}}$ is any vector, then

$$\frac{1}{|\vec{\mathbf{v}}|}\,\vec{\mathbf{v}}$$

is always a unit vector that points in the same direction as $\vec{\mathbf{v}}$.

Example:

$$\vec{\mathbf{v}} = \langle 3,\ 4 \rangle$$

$$|\vec{\mathbf{v}}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Therefore, the vector $\frac{1}{5}\vec{\mathbf{v}} = \left\langle \frac{3}{5},\ \frac{4}{5} \right\rangle$ is a unit vector

Check:

$$
\begin{aligned}
\left|\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle\right| &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\
&= \sqrt{\frac{9}{25} + \frac{16}{25}} \\
&= \sqrt{1} = 1
\end{aligned}
$$

Vector Addition

If $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$, then the *vector sum* is defined to be:

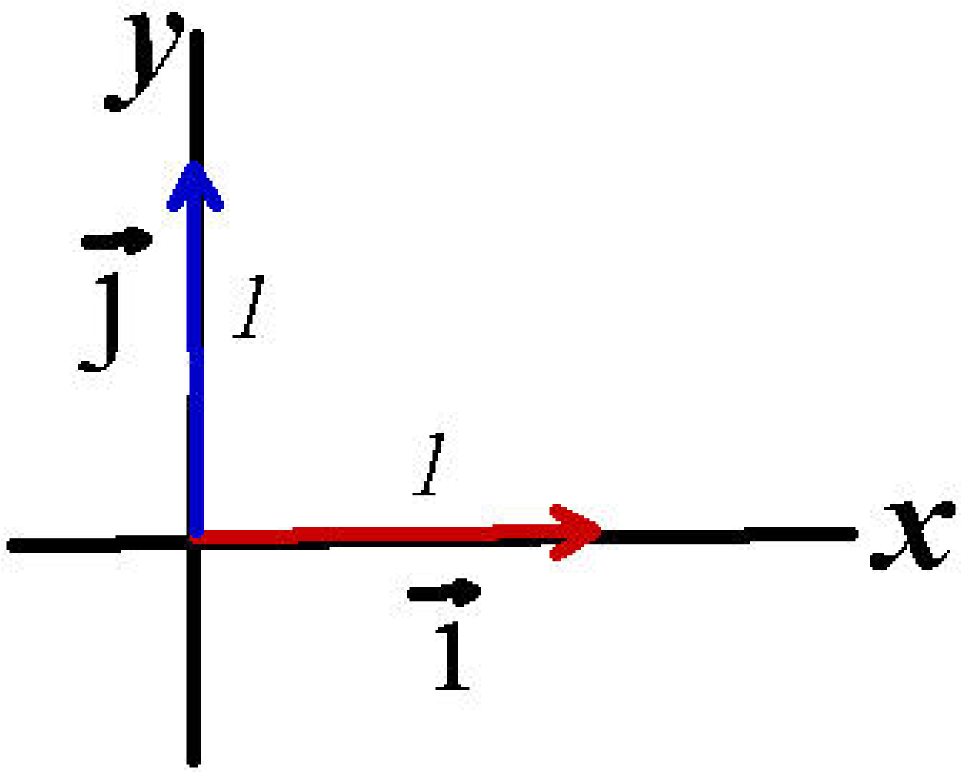
$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2 \rangle$$

If $\vec{\mathbf{u}} = \langle u_1, \, u_2, \, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, \, v_2, \, v_3 \rangle$ then

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, \, u_2 + v_2, \, u_3 + v_3 \rangle$$

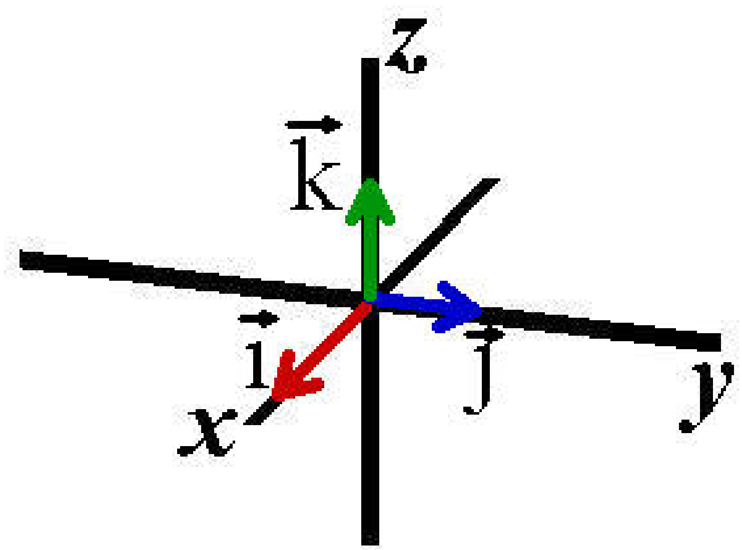
$$
\begin{aligned}
\vec{\mathbf{v}} &= \langle x,\ y\rangle \\
&= \langle x,\ 0\rangle + \langle 0,\ y\rangle
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathbf{v}} &= \langle x,\ y\rangle \\
&= \langle x,\ 0\rangle + \langle 0,\ y\rangle \\
&= x\langle 1,\ 0\rangle + y\langle 0,\ 1\rangle
\end{aligned}
$$

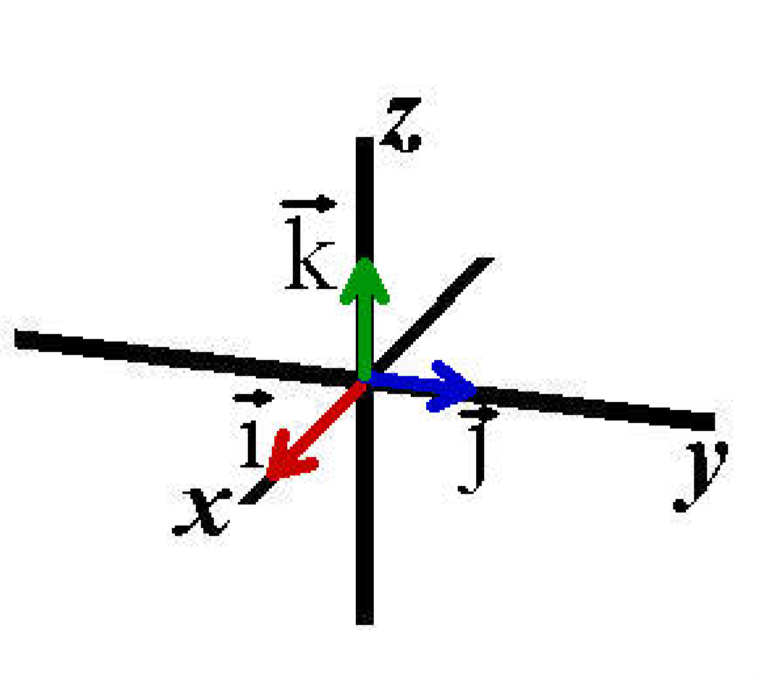

y
$\vec{j}$
1
1
x
$\vec{i}$

$$
\begin{aligned}
\vec{\mathbf{v}} &= \langle x,\ y\rangle \\
&= \langle x,\ 0\rangle + \langle 0,\ y\rangle \\
&= x\langle 1,\ 0\rangle + y\langle 0,\ 1\rangle \\
&= x\vec{\mathbf{i}} + y\vec{\mathbf{j}}
\end{aligned}
$$

$$\vec{\mathbf{i}} = \langle 1,\ 0,\ 0\rangle \qquad \vec{\mathbf{j}} = \langle 0,\ 1,\ 0\rangle \qquad \vec{\mathbf{k}} = \langle 0,\ 0,\ 1\rangle$$

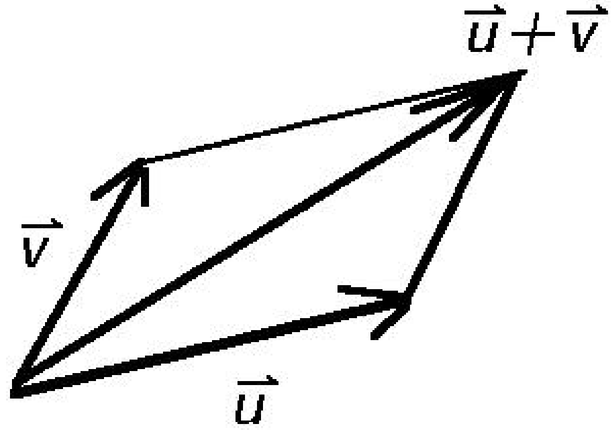


$$\vec{\mathbf{v}} = \langle x,\ y,\ z \rangle = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$$

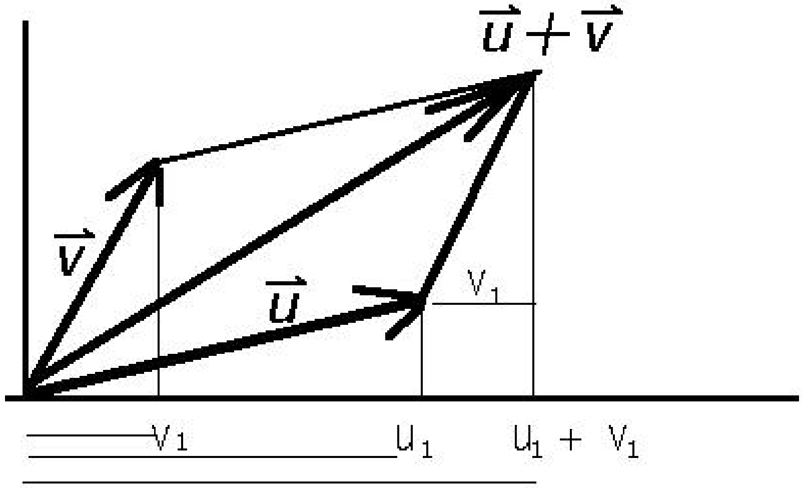


Vector Addition - Motivation

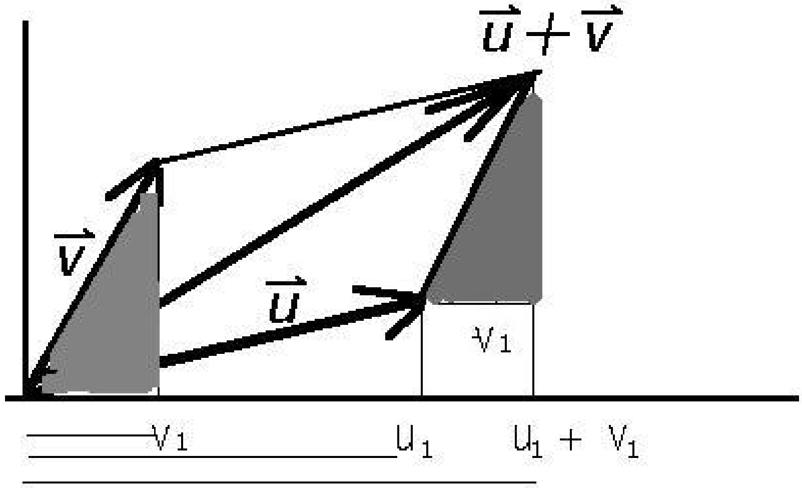
Vector addition is defined so that the vector sum is the diagonal of a parallelogram.



$$\vec{\mathbf{u}} = \langle u_1,\, u_2\rangle \qquad \vec{\mathbf{v}} = \langle v_1,\, v_2\rangle$$

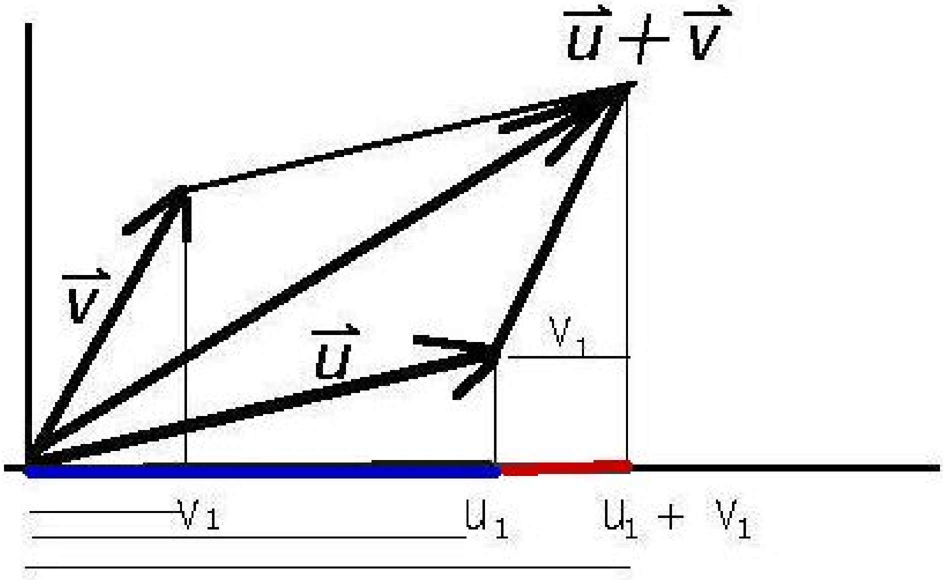


$$\vec{\mathbf{u}} = \langle u_1, u_2 \rangle \qquad \vec{\mathbf{v}} = \langle v_1, v_2 \rangle$$

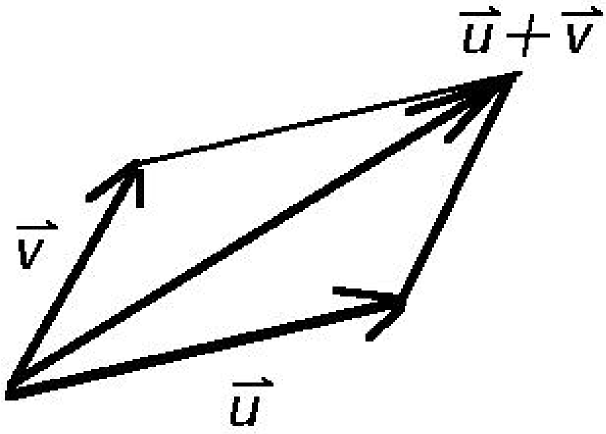


$$\vec{\mathbf{u}} = \langle u_1,\, u_2\rangle \qquad \vec{\mathbf{v}} = \langle v_1,\, v_2\rangle$$

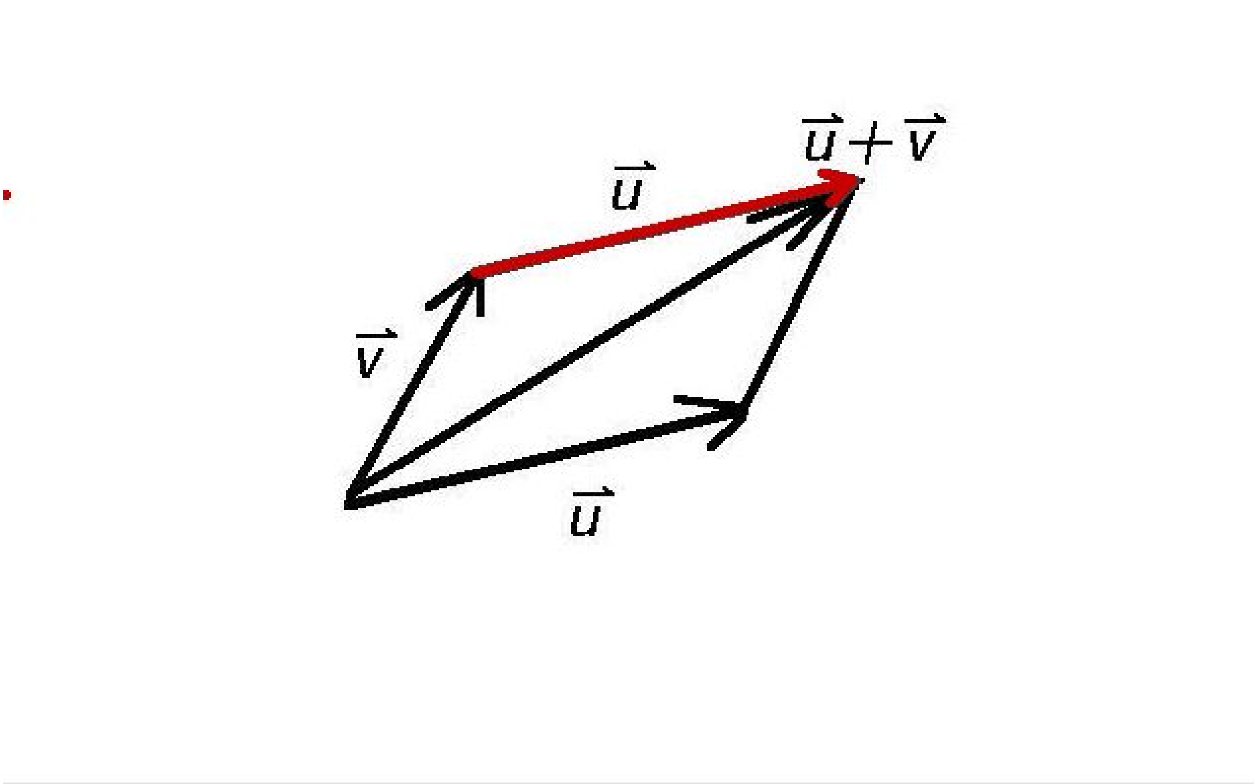
$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1,\, u_2 + v_2\rangle$$



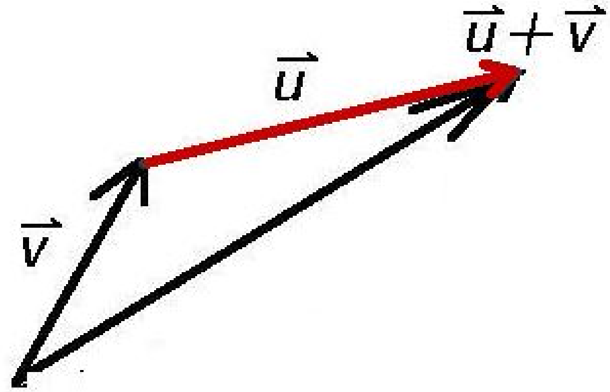
$$\vec{\mathbf{u}} + \vec{\mathbf{v}}$$



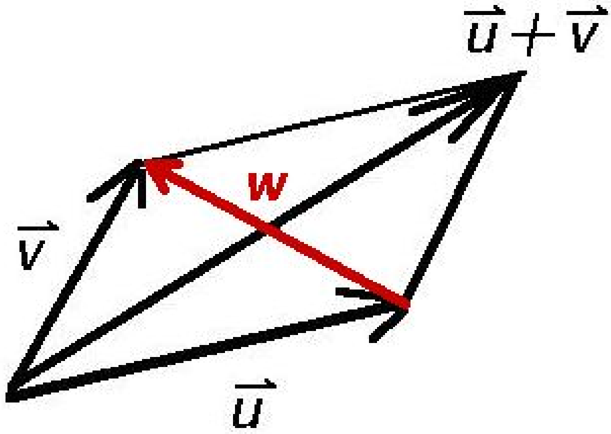
$$\vec{\mathbf{u}} + \vec{\mathbf{v}}$$



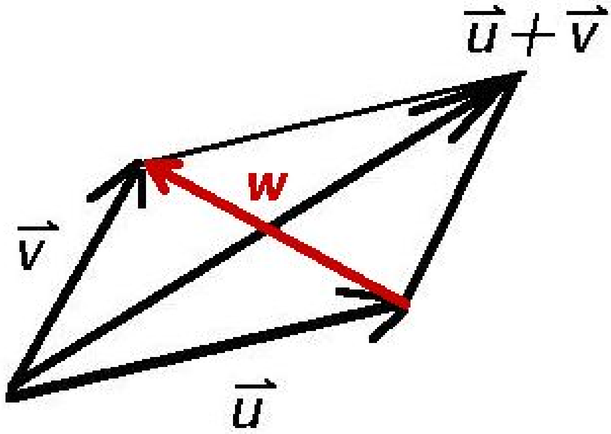
$$\vec{\mathbf{u}} + \vec{\mathbf{v}}$$



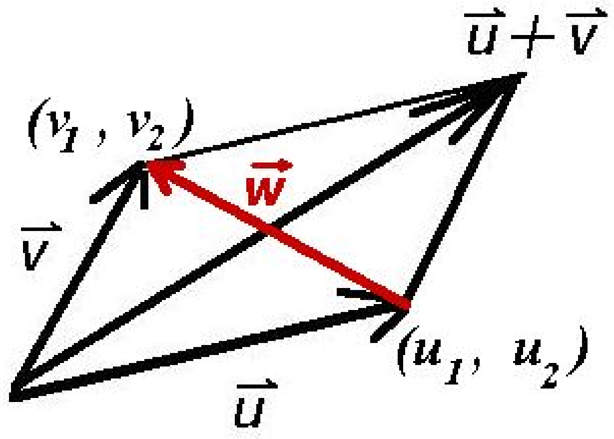
$$\vec{\mathbf{u}} + \vec{\mathbf{w}} = \vec{\mathbf{v}}$$



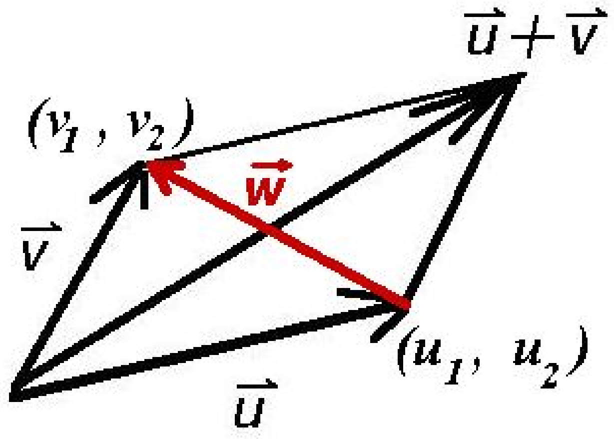
$$\text{If } \vec{\mathbf{u}} + \vec{\mathbf{w}} = \vec{\mathbf{v}} \text{ then } \vec{\mathbf{w}} = \vec{\mathbf{v}} - \vec{\mathbf{u}}$$



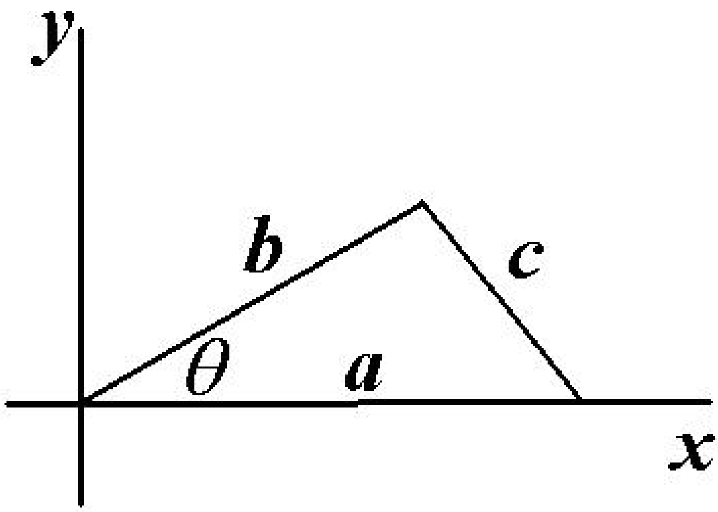
$$\vec{\mathbf{w}} = \vec{\mathbf{v}} - \vec{\mathbf{u}} = \langle v_1 - u_1,\ v_2 - u_2 \rangle$$



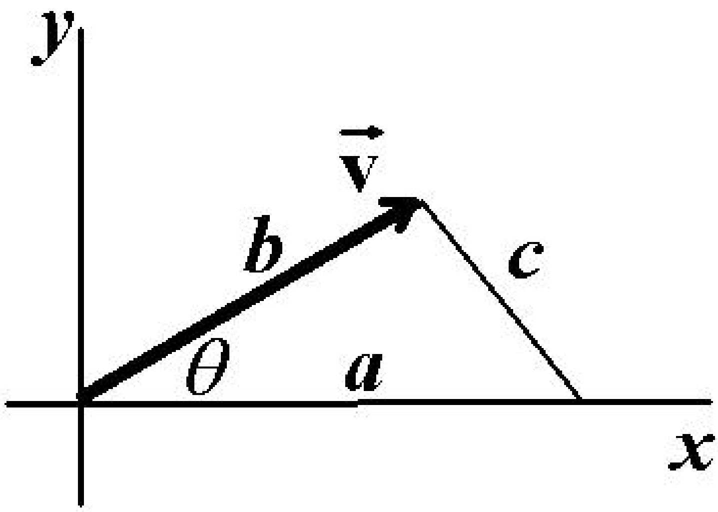
$$|\vec{\mathbf{v}} - \vec{\mathbf{u}}| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$$



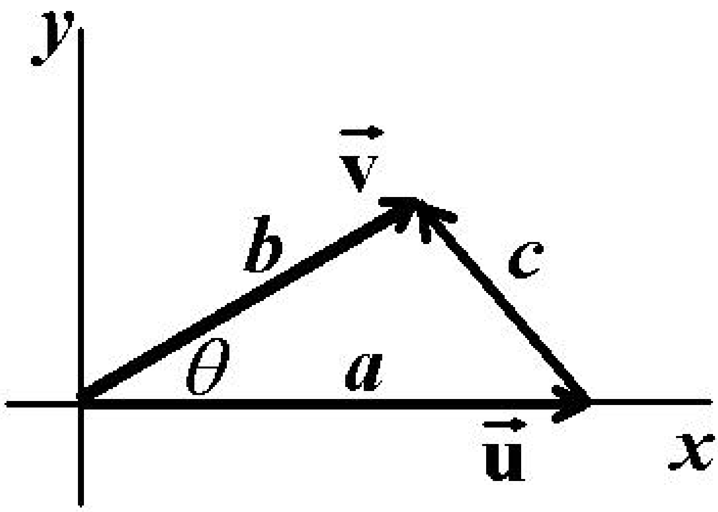
Triangle has sides a, b and c



$\vec{\mathbf{v}} = \langle b\cos\theta,\ b\sin\theta\rangle$ and $|\vec{\mathbf{v}}| = b$



$$\vec{\mathbf{v}} = \langle b\cos\theta,\ b\sin\theta\rangle \qquad \vec{\mathbf{u}} = \langle a,\ 0\rangle \qquad |\vec{\mathbf{v}} - \vec{\mathbf{u}}| = c$$



$$\vec{\mathbf{v}} = \langle b\cos\theta,\ b\sin\theta\rangle \qquad \vec{\mathbf{u}} = \langle a,\ 0\rangle \qquad |\vec{\mathbf{v}} - \vec{\mathbf{u}}| = c$$

$$|\langle b\cos\theta - a,\ b\sin\theta - 0\rangle| = c$$

$$\sqrt{(b\cos\theta - a)^2 + (b\sin\theta)^2} = c$$

$$b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2\sin^2\theta = c^2$$

$$\vec{\mathbf{v}} = \langle b\cos\theta,\ b\sin\theta\rangle \qquad \vec{\mathbf{u}} = \langle a,\ 0\rangle \qquad |\vec{\mathbf{v}} - \vec{\mathbf{u}}| = c$$

$$|\langle b\cos\theta - a,\ b\sin\theta - 0\rangle| = c$$

$$\sqrt{(b\cos\theta - a)^2 + (b\sin\theta)^2} = c$$

$$b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2\sin^2\theta = c^2$$

$$a^2 + b^2\left(\cos^2\theta + \sin^2\theta\right) - 2ab\cos\theta = c^2$$

$$|\vec{\mathbf{v}} - \vec{\mathbf{u}}| = c$$

$$|\langle b\cos\theta - a,\ b\sin\theta - 0\rangle| = c$$

$$\sqrt{(b\cos\theta - a)^2 + (b\sin\theta)^2} = c$$

$$b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2\sin^2\theta = c^2$$

$$a^2 + b^2\left(\cos^2\theta + \sin^2\theta\right) - 2ab\cos\theta = c^2$$

$$a^2 + b^2 - 2ab\cos\theta = c^2$$

The Law of Cosines $a^2 + b^2 - 2ab\cos\theta = c^2$