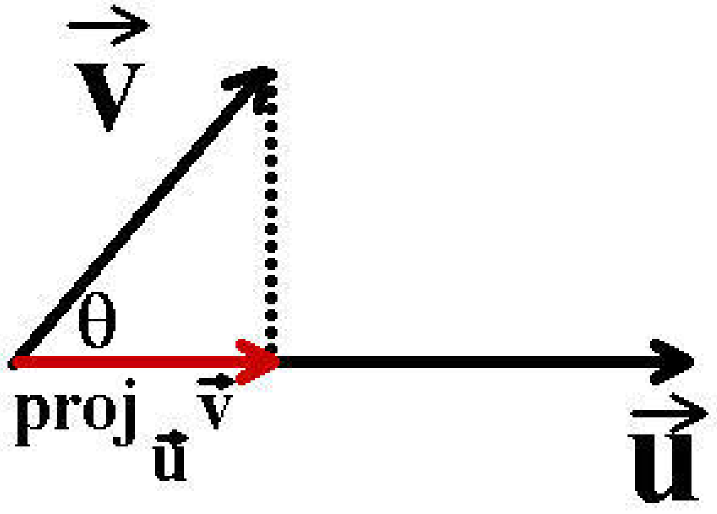
Dot Product



v
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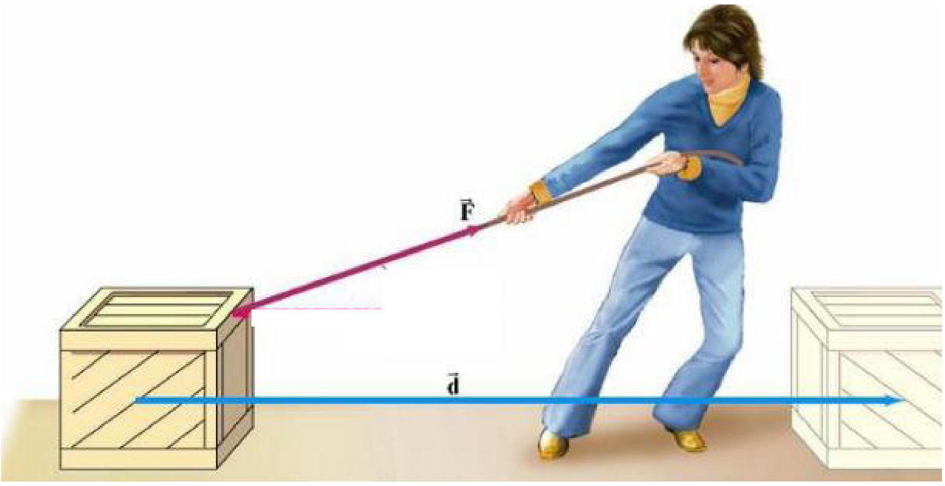
F
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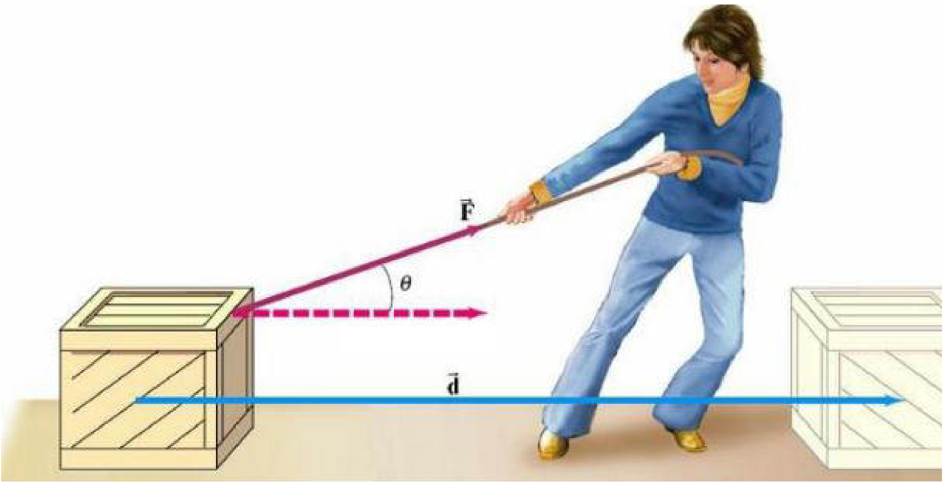
F
θ
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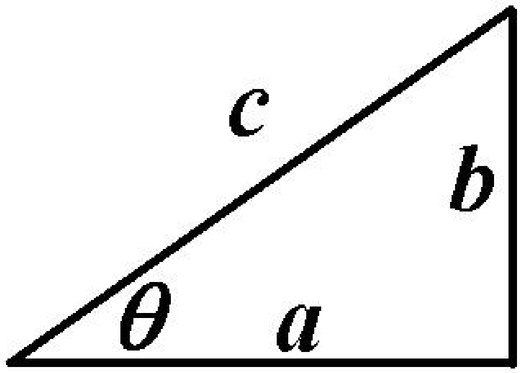
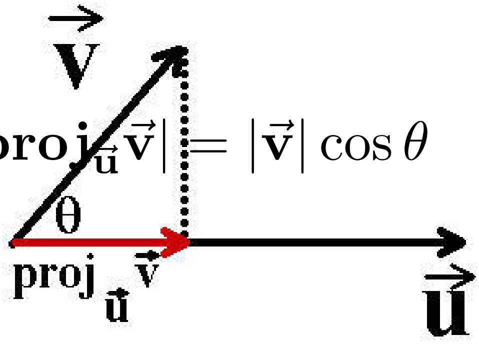
Use of cosine to find adjacent side



$$\cos\theta = \frac{a}{c} \qquad a = c\cos\theta$$

Use of cosine to find a projection



$$|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}| = |\vec{\mathbf{v}}|\cos\theta$$

Definition of Dot Product

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{u}}|\,|\vec{\mathbf{v}}| \cos\theta$$

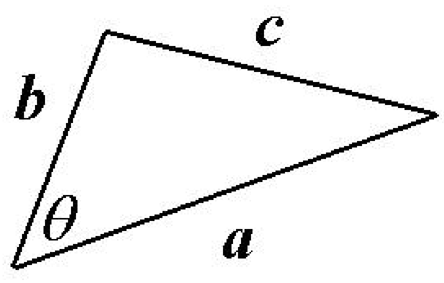
Relationship of Projection to Dot Product

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{u}}|\,|\vec{\mathbf{v}}| \cos\theta$$

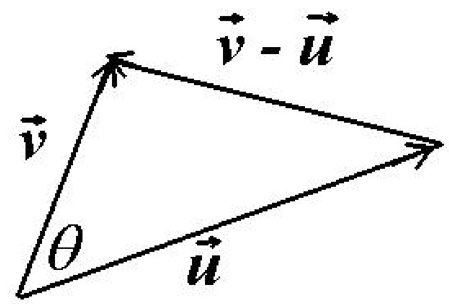
$$\frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{v}}| \cos\theta = |\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|$$

Note that $\frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}}$ is a *unit vector* in the direction of $\vec{\mathbf{u}}$.

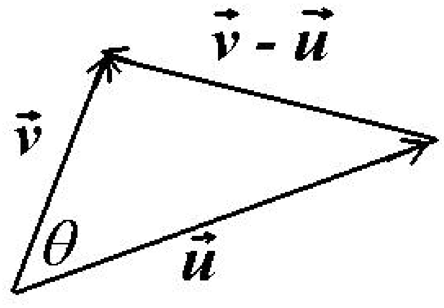
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



$$|\vec{\mathbf{v}} - \vec{\mathbf{u}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta$$



$$|\vec{\mathbf{v}} - \vec{\mathbf{u}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$



$$|\vec{\mathbf{v}} - \vec{\mathbf{u}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$\vec{\mathbf{u}} = \langle u_1,\ u_2 \rangle \qquad \vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle$$

$$\vec{\mathbf{v}} - \vec{\mathbf{u}} = \langle v_1 - u_1,\ v_2 - u_2 \rangle$$

$$(v_1 - u_1)^2 + (v_2 - u_2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$|\vec{\mathbf{v}} - \vec{\mathbf{u}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$\vec{\mathbf{u}} = \langle u_1,\ u_2 \rangle \qquad \vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle$$

$$\vec{\mathbf{v}} - \vec{\mathbf{u}} = \langle v_1 - u_1,\ v_2 - u_2 \rangle$$

$$(v_1 - u_1)^2 + (v_2 - u_2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \frac{u_1^2 + u_2^2 + v_1^2 + v_2^2 - (v_1 - u_1)^2 - (v_2 - u_2)^2}{2}$$

$$|\vec{\mathbf{v}} - \vec{\mathbf{u}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$\vec{\mathbf{u}} = \langle u_1,\ u_2 \rangle \qquad \vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle$$

$$\vec{\mathbf{v}} - \vec{\mathbf{u}} = \langle v_1 - u_1,\ v_2 - u_2 \rangle$$

$$(v_1 - u_1)^2 + (v_2 - u_2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2,\ u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2,\ v_3 \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2, \ldots, u_n \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2, \ldots, v_n \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2, \ldots, u_n \rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2, \ldots, v_n \rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \sum_{k=1}^{n} u_k v_k$$

If $\vec{\mathbf{u}} = \langle u_1,\ u_2, \ldots, u_n\rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2, \ldots, v_n\rangle$ then:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \sum_{k=1}^{n} u_k v_k$$

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{u}} = \sum_{k=1}^{n} v_k u_k$$

The dot product is commutative:

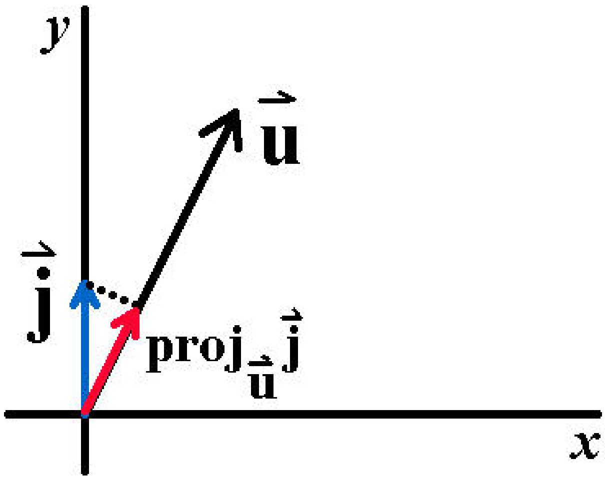
$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \vec{\mathbf{v}} \bullet \vec{\mathbf{u}}$$

$$\text{If } \vec{\mathbf{v}} = \langle v_1,\ v_2,\ v_3 \rangle \quad \text{then} \quad |\vec{\mathbf{v}}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$|\vec{\mathbf{v}}|^2 = v_1^2 + v_2^2 + v_3^2 = \langle v_1,\ v_2,\ v_3 \rangle \bullet \langle v_1,\ v_2,\ v_3 \rangle = \vec{\mathbf{v}} \bullet \vec{\mathbf{v}}$$

Example Let $\vec{\mathbf{u}} = \langle 3,\ 4\rangle$ and let $\vec{\mathbf{j}} = \langle 0,\ 1\rangle$

Calculate the length of the projection of $\vec{\mathbf{j}}$ in the direction of $\vec{\mathbf{u}}$



$$|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}}||\mathrm{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}}| = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}}||\operatorname{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}| = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$|\operatorname{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}| = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \bullet \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}}||\operatorname{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}}| = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

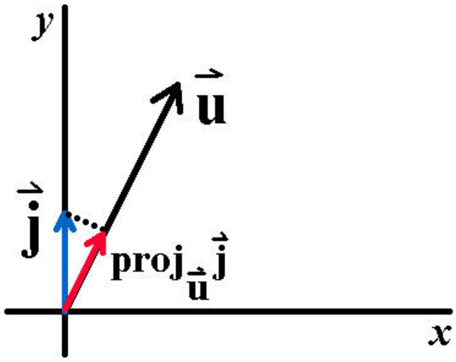
$$|\operatorname{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}}| = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \bullet \vec{\mathbf{v}}$$

If $\vec{\mathbf{v}} = \vec{\mathbf{j}} = \langle 0,\ 1 \rangle$ then:

$$|\operatorname{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{j}}| = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \bullet \vec{\mathbf{j}}$$

$$\vec{\mathbf{u}} = \langle 3,\ 4 \rangle \qquad \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} = \left\langle \frac{3}{5},\ \frac{4}{5} \right\rangle \qquad \vec{\mathbf{j}} = \langle 0,\ 1 \rangle$$

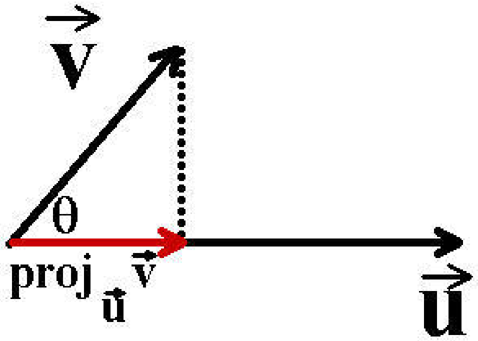
$$|\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{j}}| = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \bullet \vec{\mathbf{j}} = \langle 3/5,\ 4/5 \rangle \bullet \langle 0,\ 1 \rangle = \frac{4}{5}$$



We can use the dot product to calculate the length of $\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. How do we calculate the vector $\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ itself?

$\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ points in the same direction as $\vec{\mathbf{u}}$.

Therefore, $\dfrac{\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}}{|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|}$ is the same unit vector as $\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$



$$\frac{\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}}{|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|} = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$$

$$\frac{\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}}{|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|} = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$$

$$\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}} = |\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}}|\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$$

$$\frac{\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}}{|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|} = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$$

$$\begin{aligned}
\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}} &= |\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}}|\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \\
&= \left(\vec{\mathbf{v}} \bullet \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}\right)\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \\
&= \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}}{|\vec{\mathbf{u}}|^2}\,\vec{\mathbf{u}}
\end{aligned}$$

$$\frac{\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}}{|\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}|} = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$$

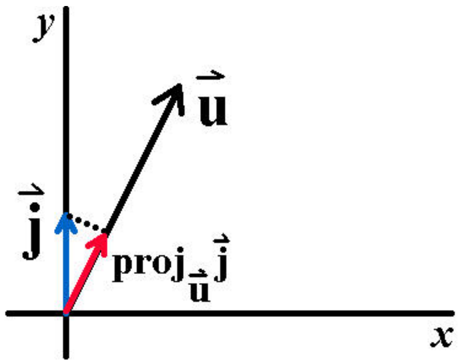
$$\begin{aligned}
\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}} &= |\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}}|\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \\
&= \left(\vec{\mathbf{v}} \bullet \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}\right)\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \\
&= \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}}{|\vec{\mathbf{u}}|^2}\,\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}\,\vec{\mathbf{u}}
\end{aligned}$$

Example Let $\vec{\mathbf{u}} = \langle 3,\ 4 \rangle$ and let $\vec{\mathbf{j}} = \langle 0,\ 1 \rangle$

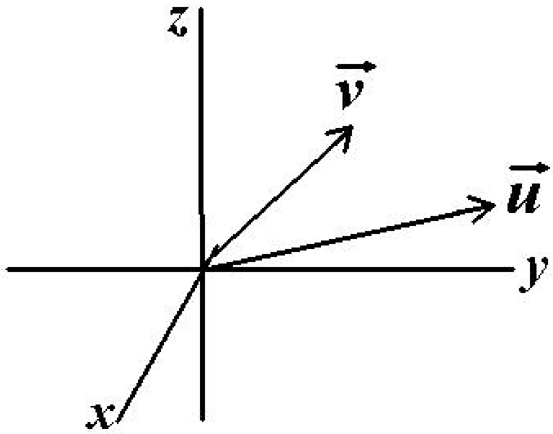
We have already know that $|\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{j}}| = \frac{4}{5}$

Now, let's calculate the vector $\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{j}}$ itself.

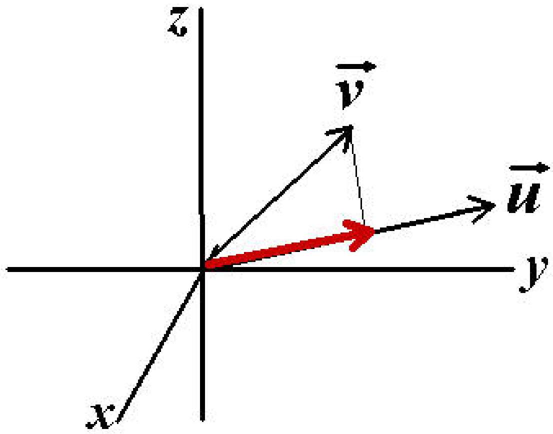
$$\mathbf{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{j}} = \frac{\vec{\mathbf{j}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}\vec{\mathbf{u}} = \frac{4}{25}\langle 3,\ 4 \rangle$$



Example: Let $\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle$ and $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1 \rangle$



Example: Calculate $|\mathbf{proj}_{\vec{\mathbf{u}}}\, \vec{\mathbf{v}}|$ and $\mathbf{proj}_{\vec{\mathbf{u}}}\, \vec{\mathbf{v}}$



$\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle$ and $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1 \rangle$

$$\begin{aligned} |\mathbf{proj}_{\vec{\mathbf{u}}}\, \vec{\mathbf{v}}| &= \vec{\mathbf{v}} \bullet \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \\ &= \langle 0,\ 1,\ 1 \rangle \bullet \frac{1}{\sqrt{6}} \langle 1,\ 2,\ 1 \rangle \end{aligned}$$

$\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle$ and $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1 \rangle$

$$\begin{aligned} |\mathbf{proj}_{\vec{\mathbf{u}}}\, \vec{\mathbf{v}}| &= \vec{\mathbf{v}} \bullet \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} \\ &= \langle 0,\ 1,\ 1 \rangle \bullet \frac{1}{\sqrt{6}} \langle 1,\ 2,\ 1 \rangle \\ &= \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2} \end{aligned}$$

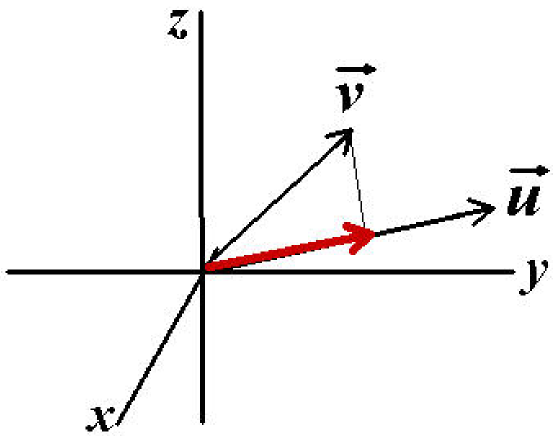
$\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle$ and $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1 \rangle$

$$\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}} = \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}\vec{\mathbf{u}}$$

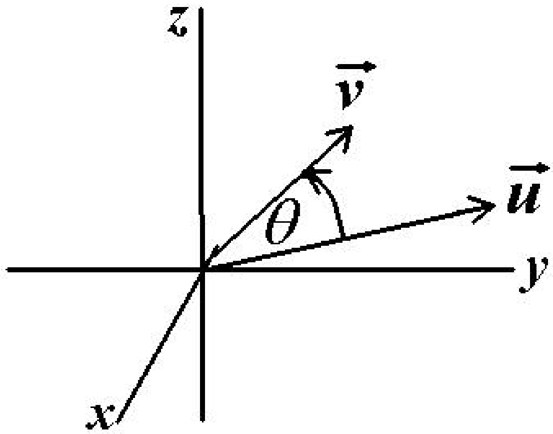
$\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle$ and $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1 \rangle$

$$\begin{aligned}
\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}} &= \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}\vec{\mathbf{u}} \\
&= \frac{\langle 0,\ 1,\ 1 \rangle \bullet \langle 1,\ 2,\ 1 \rangle}{6}\langle 1,\ 2,\ 1 \rangle \\
&= \frac{3}{6}\langle 1,\ 2,\ 1 \rangle \\
&= \frac{1}{2}\langle 1,\ 2,\ 1 \rangle
\end{aligned}$$

$$\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}} = \frac{1}{2}\vec{\mathbf{u}}$$



Example: Let $\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle$ and let $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1 \rangle$
Calculate the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$



Example: Let $\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle$ and let $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1 \rangle$
Calculate the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$

$$|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$\sqrt{6}\sqrt{2}\cos\theta = 3$$

$$\cos\theta = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

Example: Let $\vec{\mathbf{u}} = \langle 1,\ 2,\ 1\rangle$ and let $\vec{\mathbf{v}} = \langle 0,\ 1,\ 1\rangle$
Calculate the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$

$$|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$$

$$\sqrt{6}\sqrt{2}\cos\theta = 3$$

$$\cos\theta = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6} \text{ radians}$$

Example

$$\vec{\mathbf{u}} = \langle 3,\ 4 \rangle \qquad \vec{\mathbf{w}} = -\vec{\mathbf{i}} = \langle -1,\ 0 \rangle$$

Calculate the dot product

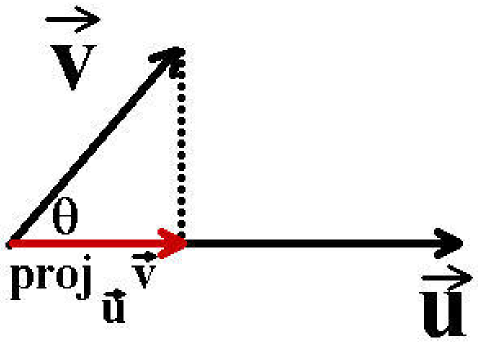
$$\vec{\mathbf{u}} \bullet \vec{\mathbf{w}} = -3$$

What does it mean when the dot product is negative?

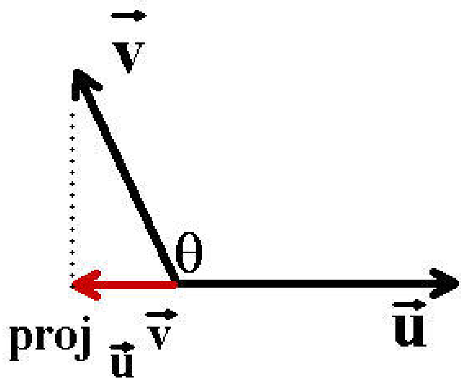
$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

The dot product is only negative when $\cos\theta$ is negative

If $0 < \theta < \frac{\pi}{2}$ then $\cos\theta > 0$ and the dot product is positive



If $\frac{\pi}{2} < \theta < \pi$ then $\cos\theta < 0$ and the dot product is negative



What does it mean if the dot product is zero?

What does it mean if the dot product is zero?

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = 0$$

$$|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = 0$$

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have nonzero length, then $\cos\theta = 0$

$$\theta = \frac{\pi}{2}$$

When two vectors are perpendicular (orthogonal) then the dot product is zero.

Example

Let $\vec{\mathbf{x}} = \langle 2,\ 1 \rangle$. Find a vector $\vec{\mathbf{y}}$ that is perpendicular to $\vec{\mathbf{x}}$

$$\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0$$

$$\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0$$

$$\langle 2,\ 1\rangle \bullet \langle y_1,\ y_2\rangle = 0$$

$$2y_1 + y_2 = 0$$

$$y_2 = -2y_1$$

$$\vec{\mathbf{y}} = \langle y_1,\ y_2\rangle = \langle y_1,\ -2y_1\rangle = y_1\langle 1,\ -2\rangle$$

Thus, any scalar multiple of $\langle 1,\ -2\rangle$ is perpendicular to $\langle 2,\ 1\rangle$

$\langle x_1,\ x_2\rangle$ is always perpendicular to $\langle x_2,\ -x_1\rangle$

The vectors $x_1\vec{\mathbf{i}} + x_2\vec{\mathbf{j}}$ and $x_2\vec{\mathbf{i}} - x_1\vec{\mathbf{j}}$ are always perpendicular to each other

Determinant notation:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If you need a vector perpendicular to $x_1\vec{\mathbf{i}} + x_2\vec{\mathbf{j}}$, calculate the determinant:

$$\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} \\ x_1 & x_2 \end{vmatrix} = x_2\vec{\mathbf{i}} - x_1\vec{\mathbf{j}}$$