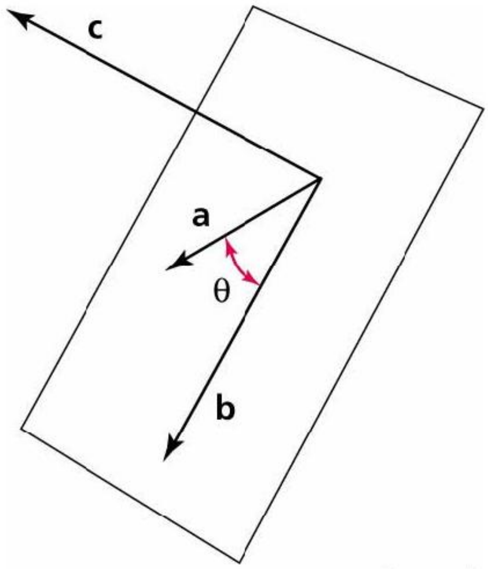
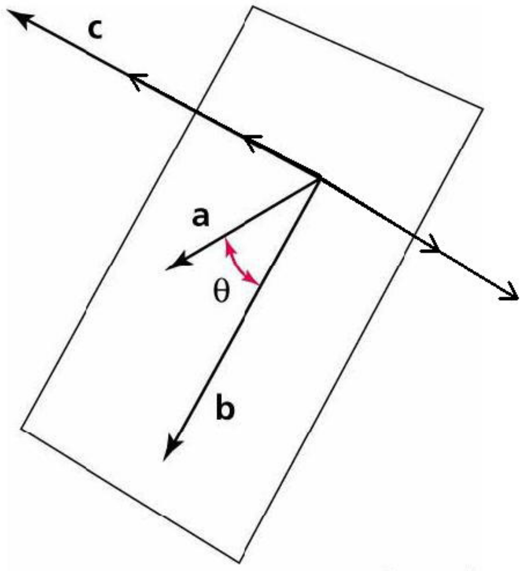
The Cross Product
Dr. Elliott Jacobs

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}}$$

Find a vector perpendicular to two given vectors



There are many vectors perpendicular to two given vectors.



Find a vector $\vec{\mathbf{w}} = \langle w_1,\ w_2,\ w_3 \rangle$ that is perpendicular to both $\vec{\mathbf{x}} = \langle x_1,\ x_2,\ x_3 \rangle$ and $\vec{\mathbf{y}} = \langle y_1,\ y_2,\ y_3 \rangle$

$$\vec{\mathbf{x}} \bullet \vec{\mathbf{w}} = 0 \qquad \vec{\mathbf{y}} \bullet \vec{\mathbf{w}} = 0$$

$$\vec{\mathbf{x}} \bullet \vec{\mathbf{w}} = 0 \qquad \vec{\mathbf{y}} \bullet \vec{\mathbf{w}} = 0$$

$$x_1 w_1 + x_2 w_2 + x_3 w_3 = 0$$

$$y_1 w_1 + y_2 w_2 + y_3 w_3 = 0$$

$$x_1 w_1 + x_2 w_2 + x_3 w_3 = 0$$
$$y_1 w_1 + y_2 w_2 + y_3 w_3 = 0$$

$$x_1 w_1 + x_2 w_2 = -x_3 w_3$$
$$y_1 w_1 + y_2 w_2 = -y_3 w_3$$

$$x_1 w_1 + x_2 w_2 = -x_3 w_3$$

$$y_1 w_1 + y_2 w_2 = -y_3 w_3$$

Solve for w_1 and w_2 in terms of w_3

$$w_1 = \frac{(x_2 y_3 - x_3 y_2)}{(x_1 y_2 - y_1 x_2)} w_3$$

$$w_2 = \frac{(x_1 y_3 - x_3 y_1)}{(y_1 x_2 - x_1 y_2)} w_3$$

$$x_1 w_1 + x_2 w_2 = -x_3 w_3$$

$$y_1 w_1 + y_2 w_2 = -y_3 w_3$$

Solve for w_1 and w_2 in terms of w_3

$$w_1 = \frac{(x_2 y_3 - x_3 y_2)}{(x_1 y_2 - y_1 x_2)} w_3$$

$$w_2 = -\frac{(x_1 y_3 - x_3 y_1)}{(x_1 y_2 - y_1 x_2)} w_3$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$w_1 = \frac{(x_2 y_3 - x_3 y_2)}{(x_1 y_2 - y_1 x_2)} w_3 = \frac{\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} w_3$$

$$w_2 = -\frac{(x_1 y_3 - x_3 y_1)}{(x_1 y_2 - y_1 x_2)} w_3 = -\frac{\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} w_3$$

$$w_1 = \frac{\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} w_3 \qquad w_2 = -\frac{\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} w_3$$

For simplicity, take $\quad w_3 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

$$w_1 = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \qquad w_2 = -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \qquad w_3 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$\begin{aligned}\vec{\mathbf{w}} &= w_1\vec{\mathbf{i}} + w_2\vec{\mathbf{j}} + w_3\vec{\mathbf{k}} \\ &= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\vec{\mathbf{k}}\end{aligned}$$

$$
\begin{aligned}
\vec{\mathbf{w}} &= w_1\vec{\mathbf{i}} + w_2\vec{\mathbf{j}} + w_3\vec{\mathbf{k}} \\
&= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\vec{\mathbf{k}} \\
&= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
&= \vec{\mathbf{x}} \times \vec{\mathbf{y}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathbf{w}} &= w_1\vec{\mathbf{i}} + w_2\vec{\mathbf{j}} + w_3\vec{\mathbf{k}} \\
&= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\vec{\mathbf{k}} \\
&= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
&= \vec{\mathbf{x}} \times \vec{\mathbf{y}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathbf{w}} &= w_1\vec{\mathbf{i}} + w_2\vec{\mathbf{j}} + w_3\vec{\mathbf{k}} \\
&= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\vec{\mathbf{k}} \\
&= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
&= \vec{\mathbf{x}} \times \vec{\mathbf{y}}
\end{aligned}
$$

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\vec{\mathbf{w}} &= w_1\vec{\mathbf{i}} + w_2\vec{\mathbf{j}} + w_3\vec{\mathbf{k}} \\
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&= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
&= \vec{\mathbf{x}} \times \vec{\mathbf{y}}
\end{aligned}$$

$$
\begin{aligned}
\vec{\mathbf{w}} &= w_1\vec{\mathbf{i}} + w_2\vec{\mathbf{j}} + w_3\vec{\mathbf{k}} \\
&= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\vec{\mathbf{k}} \\
&= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
&= \vec{\mathbf{x}} \times \vec{\mathbf{y}}
\end{aligned}
$$

Example: Find a vector that is perpendicular to both of the following vectors:

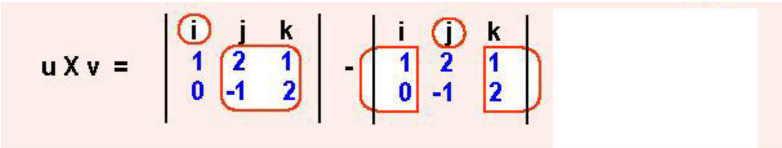
$$\vec{\mathbf{u}} = \langle 1,\ 2,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle 0,\ -1,\ 2 \rangle$$

Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \vec{\mathbf{i}}$$

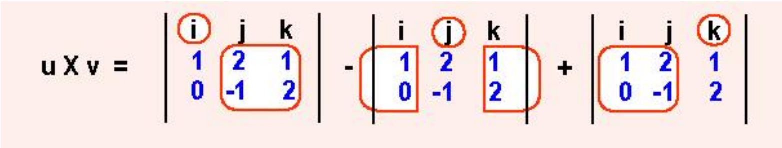
Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$



$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \vec{\mathbf{j}}$$

Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

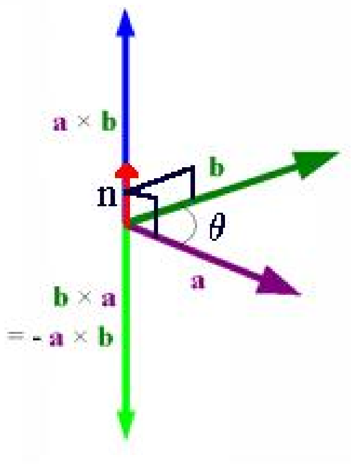


$$
\begin{aligned}
\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \vec{\mathbf{k}} \\
&= 5\vec{\mathbf{i}} - 2\vec{\mathbf{j}} - \vec{\mathbf{k}} \\
&= \langle 5,\ -2,\ -1 \rangle
\end{aligned}
$$

Compare with:

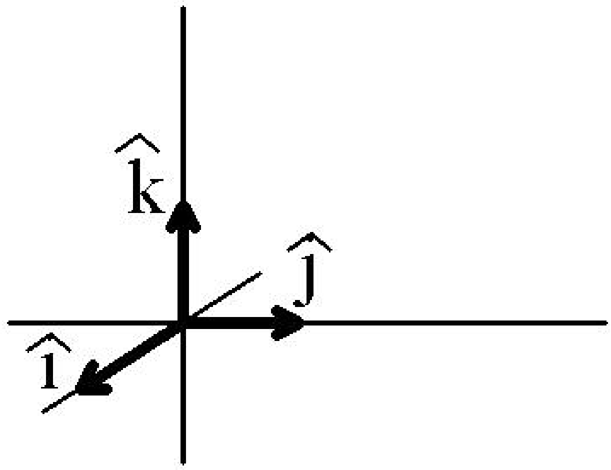
$$\vec{\mathbf{v}} \times \vec{\mathbf{u}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \langle -5,\ 2,\ 1 \rangle$$

The cross product is not commutative

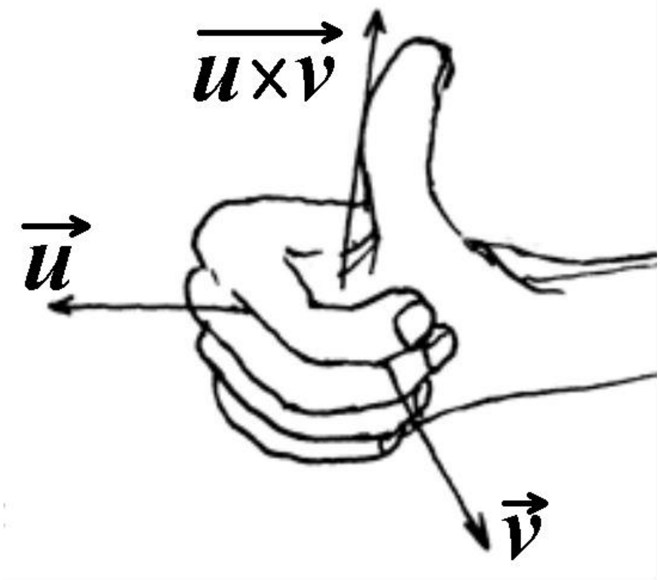


$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{\mathbf{k}} = \vec{\mathbf{k}}$$

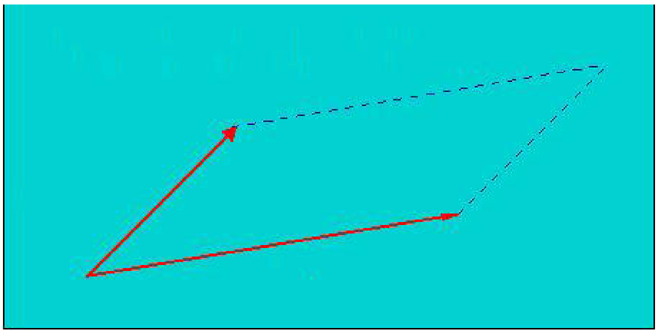
$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}} \qquad \vec{\mathbf{j}} \times \vec{\mathbf{k}} = \vec{\mathbf{i}} \qquad \vec{\mathbf{k}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}}$$



The Right Hand Rule

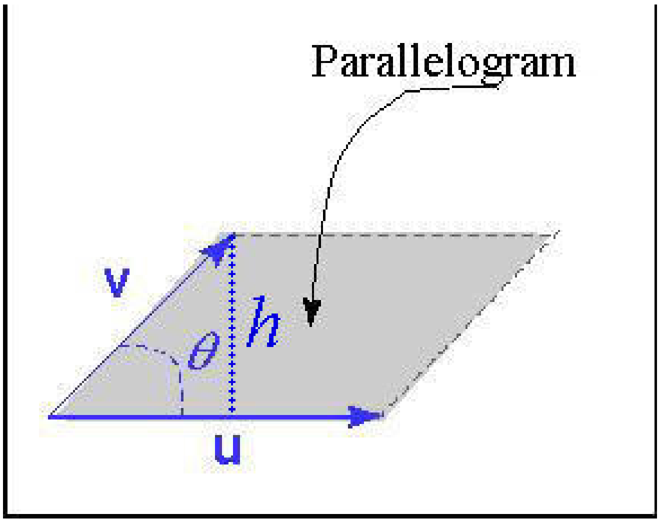


Two vectors form the edges of a parallelogram

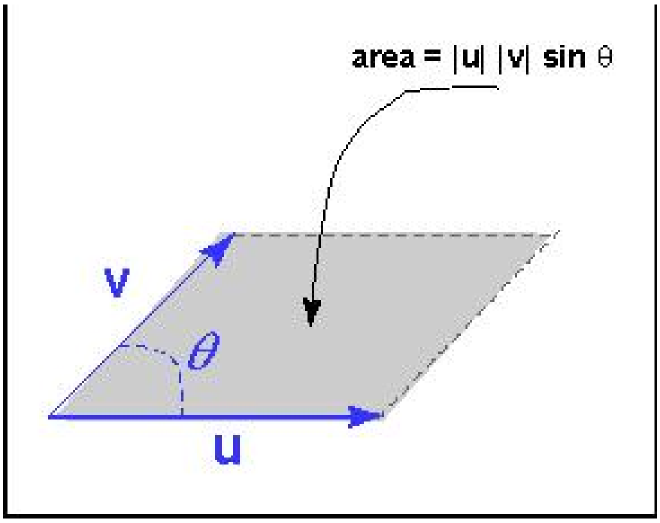


$$\text{Area} = h|\vec{\mathbf{u}}|$$

$$h = |\vec{\mathbf{v}}| \sin\theta$$

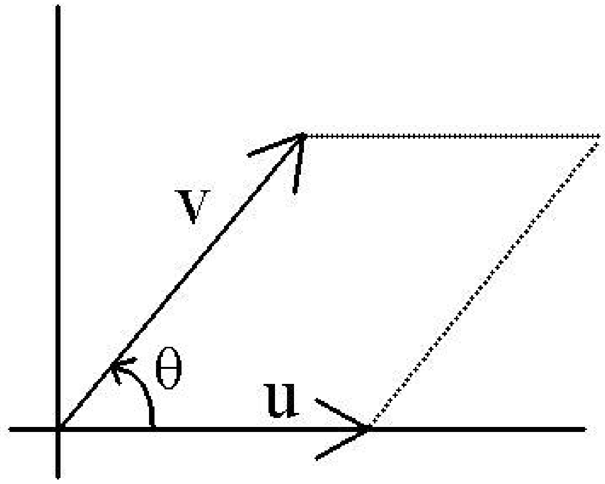


$$\text{Area} = h|\vec{\mathbf{u}}| = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|\sin\theta$$



Suppose $|\vec{\mathbf{u}}| = a$ and $|\vec{\mathbf{v}}| = b$

$$\vec{\mathbf{u}} = \langle a,\ 0,\ 0\rangle \qquad \vec{\mathbf{v}} = \langle b\cos\theta,\ b\sin\theta, 0\rangle$$



Suppose $|\vec{\mathbf{u}}| = a$ and $|\vec{\mathbf{v}}| = b$

$$\vec{\mathbf{u}} = \langle a,\ 0,\ 0\rangle \qquad \vec{\mathbf{v}} = \langle b\cos\theta,\ b\sin\theta, 0\rangle$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a & 0 & 0 \\ b\cos\theta & b\sin\theta & 0 \end{vmatrix} = (ab\sin\theta)\vec{\mathbf{k}} = (|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\sin\theta)\vec{\mathbf{k}}$$

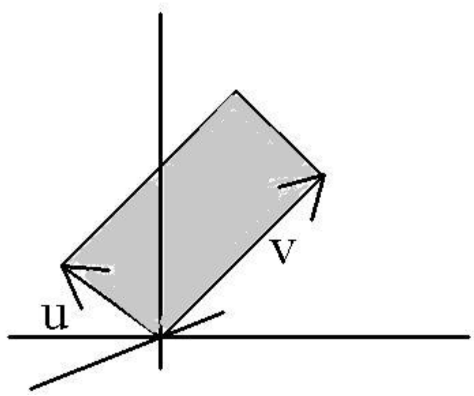
Suppose $|\vec{\mathbf{u}}| = a$ and $|\vec{\mathbf{v}}| = b$

$$\vec{\mathbf{u}} = \langle a,\ 0,\ 0\rangle \qquad \vec{\mathbf{v}} = \langle b\cos\theta,\ b\sin\theta, 0\rangle$$

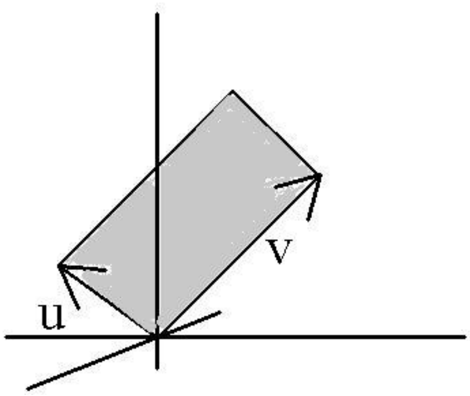
$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a & 0 & 0 \\ b\cos\theta & b\sin\theta & 0 \end{vmatrix} = (ab\sin\theta)\vec{\mathbf{k}} = (|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\sin\theta)\vec{\mathbf{k}}$$

$$|\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|\sin\theta = \textbf{Area of Parallelogram}$$

$\vec{\mathbf{u}} = \langle 1, 0, 1 \rangle$ and $\vec{\mathbf{v}} = \langle 0, 2, 2 \rangle$. Find the parallelogram area



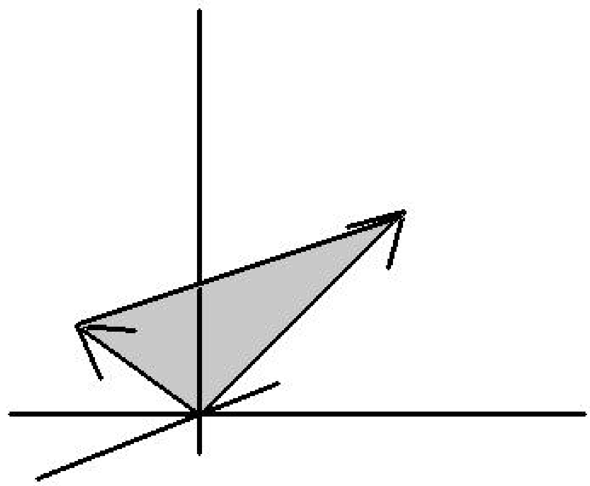
$\vec{\mathbf{u}} = \langle 1, 0, 1 \rangle$ and $\vec{\mathbf{v}} = \langle 0, 2, 2 \rangle$. Find the parallelogram area



$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = -2\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$$

$$|\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Find the area of the triangle



$\vec{\mathbf{u}} = \langle 1, 0, 1 \rangle \qquad \vec{\mathbf{v}} = \langle 0, 2, 2 \rangle$

$$\text{Area of Triangle} = \frac{1}{2}(\text{Area of parallelogram}) = \frac{1}{2} \cdot 2\sqrt{3} = \sqrt{3}$$