Integrals of Velocity, Speed and Acceleration

$$\int_a^b \vec{\mathbf{a}}\,dt \qquad\qquad \int_a^b \vec{\mathbf{v}}\,dt$$

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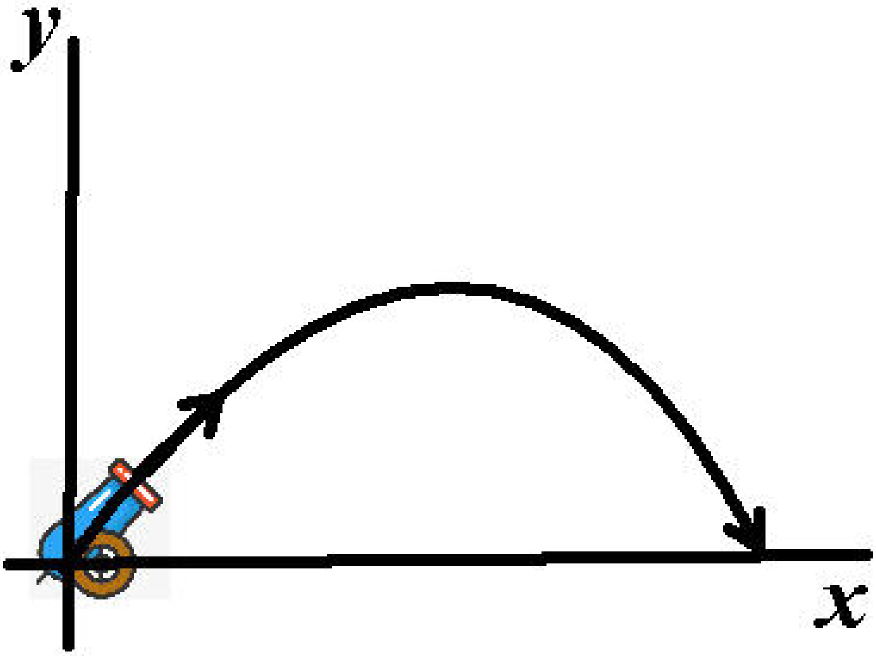
$$\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{a}} \qquad \frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{v}}$$

$$\vec{\mathbf{f}}(t) = \langle x(t),\ y(t),\ z(t) \rangle$$

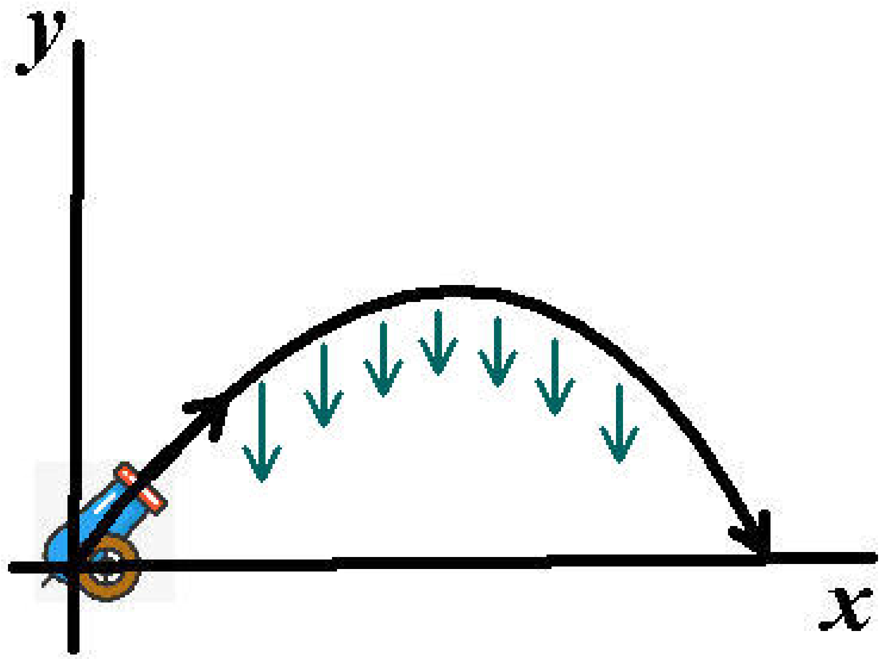
$$\vec{\mathbf{f}}'(t) = \langle x'(t),\ y'(t),\ z'(t) \rangle$$

$$\int \vec{\mathbf{f}}(t)\, dt = \left\langle \int x(t)\, dt,\ \int y(t)\, dt,\ \int z(t)\, dt \right\rangle$$

Let $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$ be the position of a projectile at time t
$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}$ and $\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}$
Assume that at $t = 0$, $\vec{\mathbf{r}} = \langle 0,\ 0\rangle$ and $|\vec{\mathbf{v}}| = 8$
If initial angle is $\frac{\pi}{4}$, then $\vec{\mathbf{v}} = \langle 8\cos\frac{\pi}{4},\ 8\sin\frac{\pi}{4}\rangle = \langle 4\sqrt{2},\ 4\sqrt{2}\rangle$



$$\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{a}} = \langle 0,\ -g \rangle$$



$$\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{a}} = \langle 0,\ -g \rangle$$

$$\vec{\mathbf{v}} = \int \vec{\mathbf{a}}\, dt = \left\langle \int 0\, dt,\ \int -g\, dt \right\rangle$$

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$$\vec{\mathbf{v}} = \int \vec{\mathbf{a}}\, dt = \left\langle \int 0\, dt,\ \int -g\, dt \right\rangle$$

$$\vec{\mathbf{v}} = \langle C_1,\ -gt + C_2 \rangle$$

If $\vec{\mathbf{v}} = \langle 4\sqrt{2},\ 4\sqrt{2} \rangle$ at $t = 0$ then

$$\langle 4\sqrt{2},\ 4\sqrt{2} \rangle = \langle C_1,\ C_2 \rangle$$

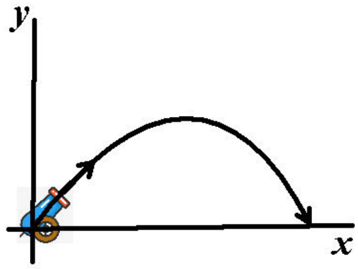
$$C_1 = C_2 = 4\sqrt{2}$$

If $\vec{\mathbf{v}} = \langle C_1,\ -gt + C_2\rangle = \langle 4\sqrt{2},\ -gt + 4\sqrt{2}\rangle$ then:

$$\vec{\mathbf{r}} = \int \vec{\mathbf{v}}\,dt = \langle 4\sqrt{2}\,t + k_1,\ -\frac{1}{2}gt^2 + 4\sqrt{2}\,t + k_2\rangle$$

If $\vec{\mathbf{r}} = \langle 0,\ 0\rangle$ when $t = 0$ then $k_1 = k_2 = 0$

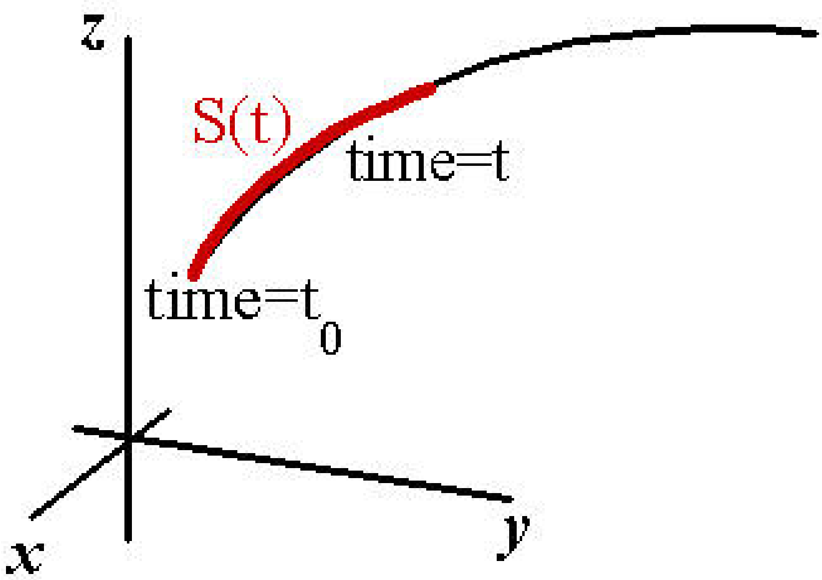
$$\vec{\mathbf{r}} = \langle 4\sqrt{2}\,t,\ -\frac{1}{2}gt^2 + 4\sqrt{2}\,t\rangle$$



Integral of Speed

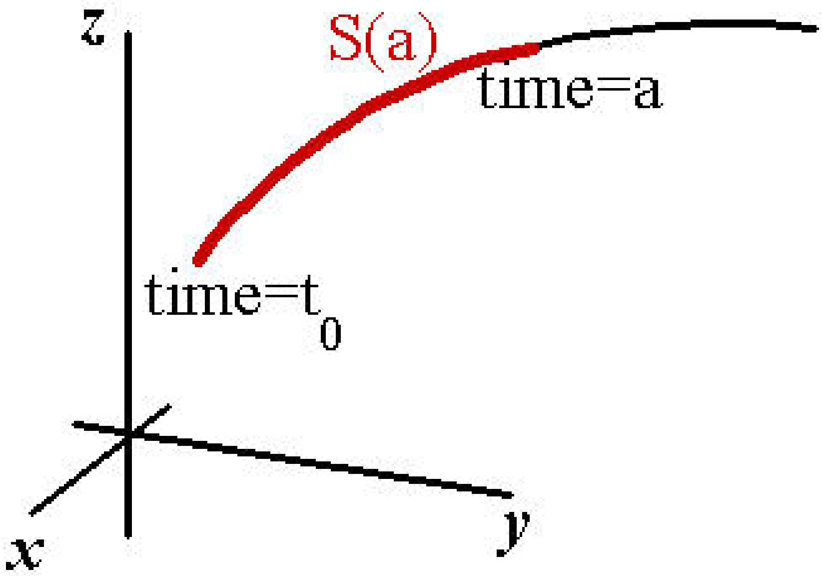
$S(t)$ is the distance traveled between time$= t_0$ and time$= t$.

$$|\vec{\mathbf{v}}| = S'(t)$$



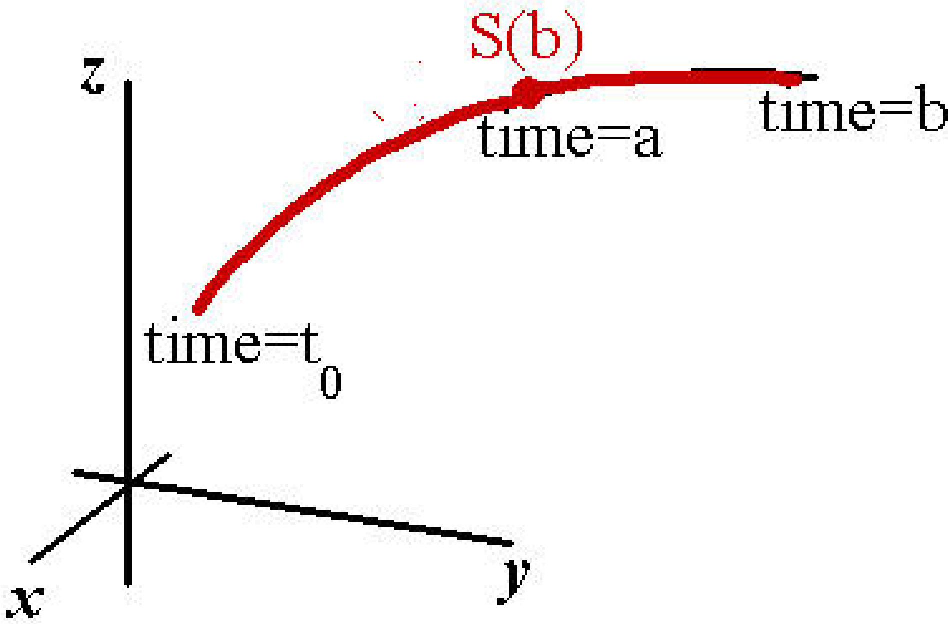
Integral of Speed

$S(a)$ is the distance traveled between time= t_0 and time= a.



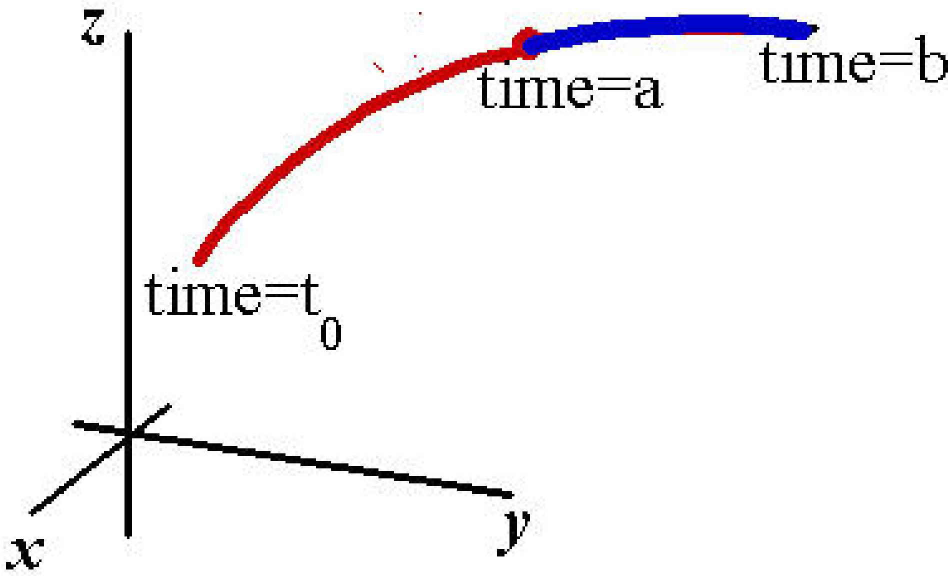
Integral of Speed

$S(b)$ is the distance traveled between time$= t_0$ and time$= b$.



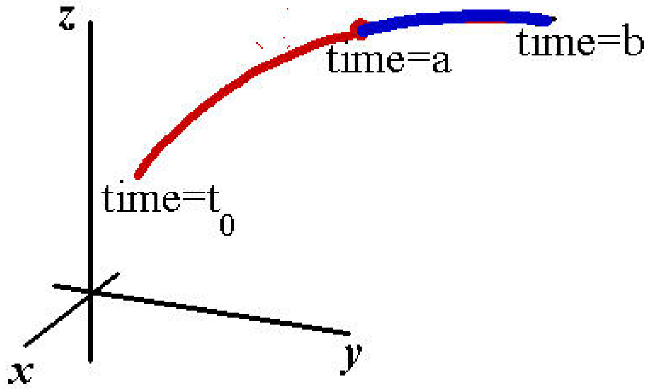
Distance along the blue segment

$$s = S(b) - S(a)$$



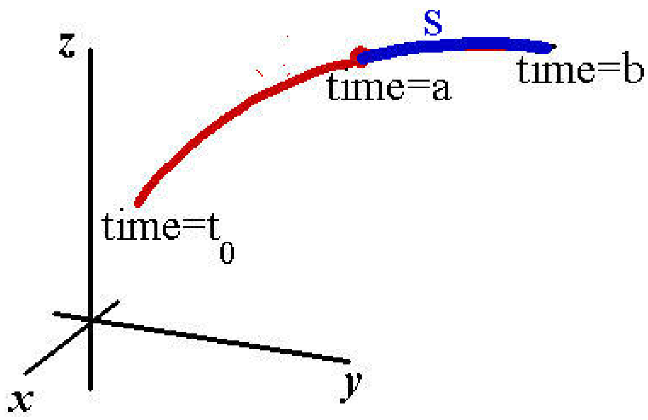
$$|\vec{\mathbf{v}}| = S'(t)$$

$$\int_a^b |\vec{\mathbf{v}}|\,dt = \int_a^b S'(t)\,dt = S(b) - S(a)$$



Arc length formula:

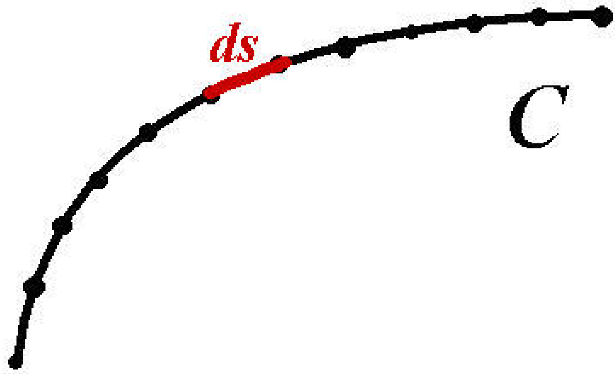
$$s = \int_a^b |\vec{\mathbf{v}}|\, dt$$



$$\text{distance} = (\text{speed})(\text{time})$$

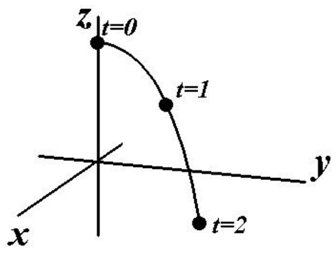
$$ds = |\vec{\mathbf{v}}|\,dt$$

$$\int_C ds = \int |\vec{\mathbf{v}}|\,dt$$



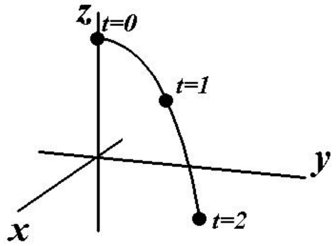
$$\vec{\mathbf{r}} = \langle t,\ t,\ 4 - t^2 \rangle \qquad \vec{\mathbf{v}} = \langle 1,\ 1,\ -2t \rangle \qquad |\vec{\mathbf{v}}| = \sqrt{2 + 4t^2}$$

$$s = \int_0^2 \sqrt{2 + 4t^2}\, dt$$



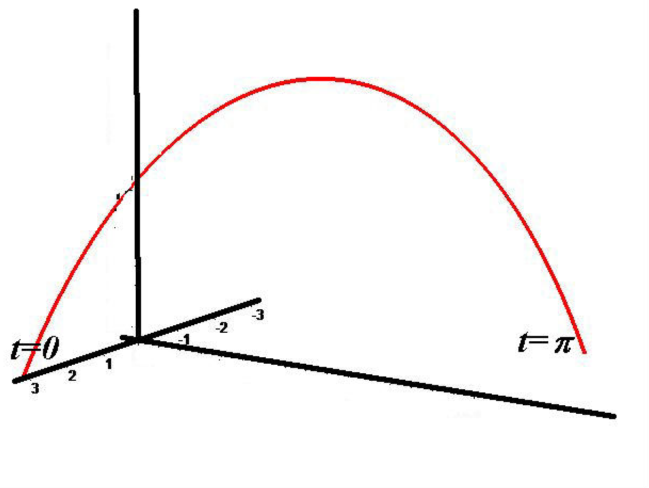
$$\vec{\mathbf{r}} = \langle t,\ t,\ 4 - t^2\rangle \qquad \vec{\mathbf{v}} = \langle 1,\ 1,\ -2t\rangle \qquad |\vec{\mathbf{v}}| = \sqrt{2 + 4t^2}$$

$$s = \int_0^2 \sqrt{2 + 4t^2}\, dt = 3\sqrt{2} + \frac{1}{2}\ln(2\sqrt{2} + 3)$$



$$\vec{\mathbf{r}} = \langle \pi \cos t,\ t,\ \pi \sin t \rangle \qquad \vec{\mathbf{v}} = \langle -\pi \sin t,\ 1,\ \pi \cos t \rangle$$

Find the length of the curve for $0 \leq t \leq \pi$



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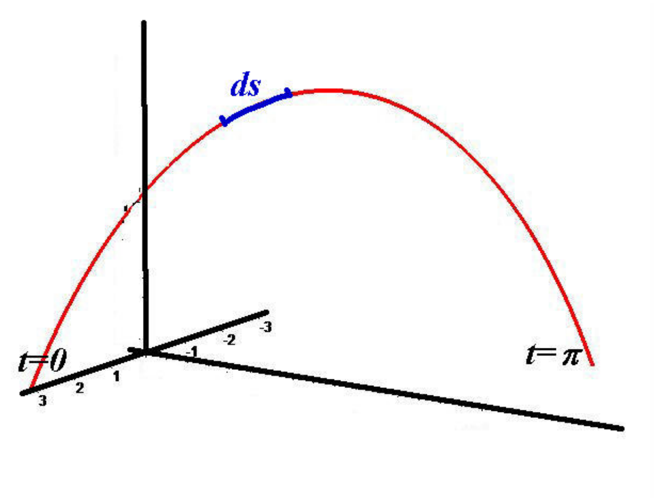
Find the length of the curve for $0 \le t \le \pi$

$$|\vec{\mathbf{v}}| = \sqrt{\pi^2 \sin^2 t + 1 + \pi^2 \cos^2 t} = \sqrt{\pi^2 + 1}$$

$$s = \int_0^{\pi} \sqrt{\pi^2 + 1}\, dt = \Big[\sqrt{\pi^2 + 1}\, t\Big]_0^{\pi} = \pi\sqrt{\pi^2 + 1}$$

$$\vec{\mathbf{r}} = \langle \pi \cos t,\ t,\ \pi \sin t \rangle \qquad \vec{\mathbf{v}} = \langle -\pi \sin t,\ 1,\ \pi \cos t \rangle$$

Suppose electric charge is distributed along this path with a charge density of: $f(x, y, z) = \frac{1}{x^2+y^2+z^2}$ coulombs per meter The amount of charge on one segment of length ds would be $f(x, y, z)\, ds$ coulombs.



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Find the total amount of charge along this curve.

$$Q = \int_C f(x, y, z)\,ds$$

$$\vec{\mathbf{r}} = \langle \pi \cos t,\ \ t,\ \ \pi \sin t \rangle \qquad \vec{\mathbf{v}} = \langle -\pi \sin t,\ \ 1,\ \ \pi \cos t \rangle$$

$$ds = |\vec{\mathbf{v}}|\, dt = \sqrt{\pi^2 + 1}\, dt$$

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2} = \frac{1}{\pi^2 + t^2}$$

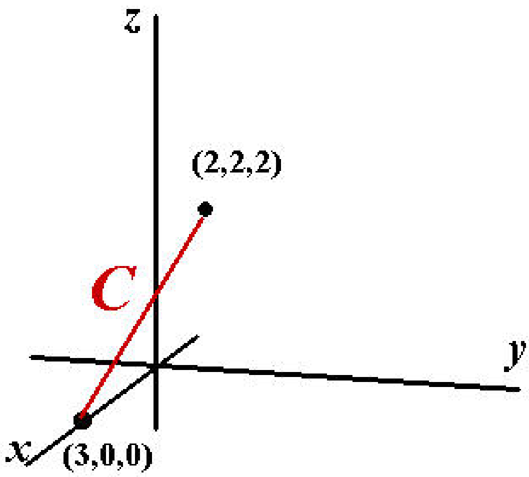
$$Q = \int_C f(x, y, z)\, ds = \int_0^{\pi} \frac{\sqrt{\pi^2 + 1}}{\pi^2 + t^2}\, dt$$

$$
\begin{aligned}
Q &= \int_C f(x,y,z)\,ds \\
&= \int_0^{\pi} \frac{\sqrt{\pi^2+1}}{\pi^2+t^2}\,dt \\
&= \sqrt{\pi^2+1}\int_0^{\pi} \frac{1}{\pi^2+t^2}\,dt \\
&= \frac{1}{4}\sqrt{\pi^2+1}
\end{aligned}
$$

If $f(x, y, z)$ is charge density in coulombs/meter then the total amount of charge along a curve C is $\int_C f(x, y, z)\, ds$

If $f(x, y, z)$ is mass density in kilograms/meter then the total amount of mass along a curve C is $\int_C f(x, y, z)\, ds$

Let C be the line segment from $(3,\ 0,\ 0)$ to $(2,\ 2,\ 2)$
Let $f(x, y, z) = 2x + y + z^2$.
Calculate the line integral $\int_C f(x, y, z)\, ds$



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$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}} = \langle 3,\ 0,\ 0\rangle + t\langle -1,\ 2,\ 2\rangle = \langle 3 - t,\ 2t,\ 2t\rangle$$

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$$\vec{\mathbf{v}} = \langle -1,\ 2,\ 2\rangle \qquad |\vec{\mathbf{v}}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$$

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$$ds = |\vec{\mathbf{v}}|\, dt = 3\, dt$$

If $\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle 3 - t,\ 2t,\ 2t\rangle$ then:

$$x = 3 - t \qquad y = 2t \qquad z = 2t$$

$$f(x, y, z) = 2x + y + z^2 = 2(3 - t) + 2t + (2t)^2 = 6 + 4t^2$$

$$f(x,y,z)\,ds = (6+4t^2)(3\,dt) = (18+12t^2)\,dt$$

$$\int_C f(x,y,z)\,ds = \int_0^1 (18+12t^2)\,dt = 22$$