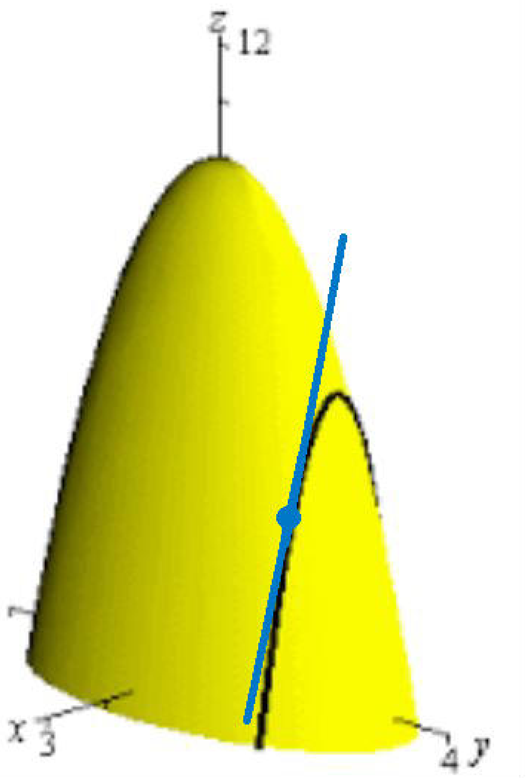
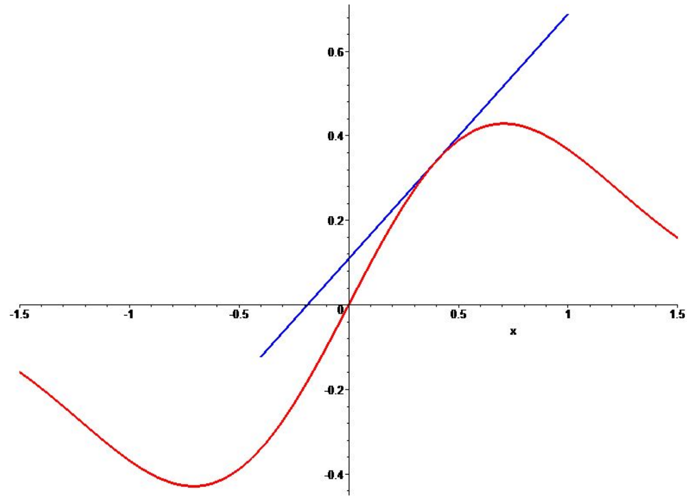
# Introduction to Partial Derivatives



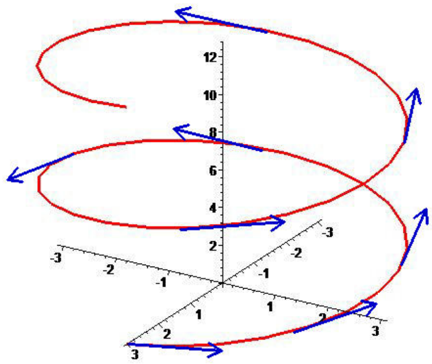
$$y = f(x)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

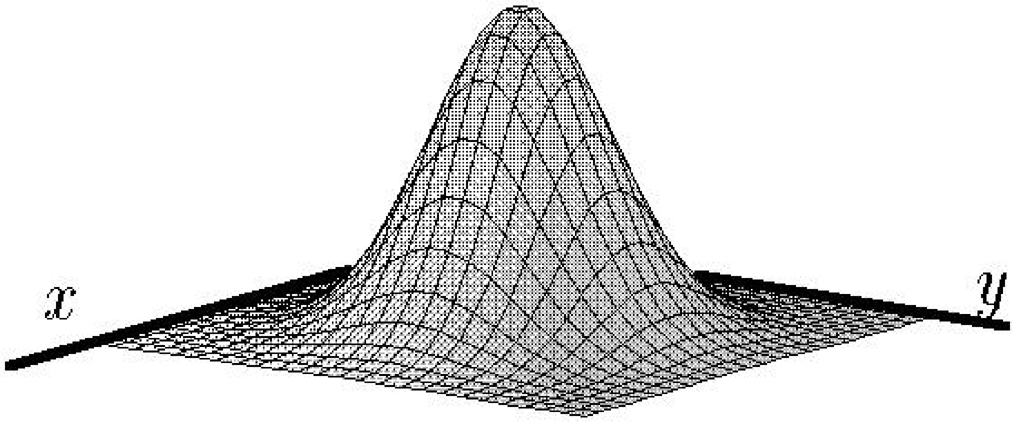


$$\vec{\mathbf{f}}(t) = \langle x(t),\ y(t),\ z(t)\rangle$$

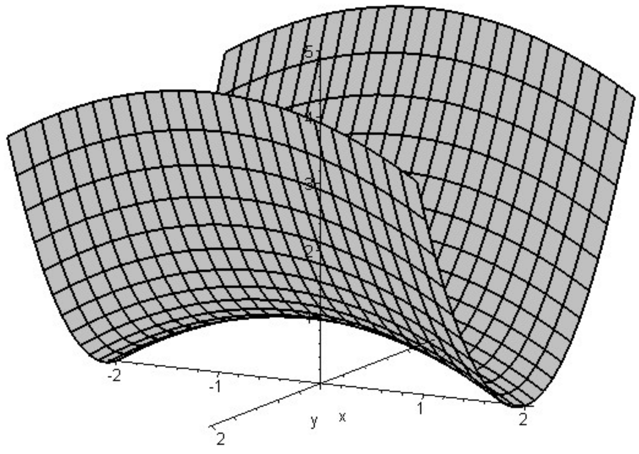
$$\vec{\mathbf{f}}'(t) = \lim_{h\to 0} \frac{\vec{\mathbf{f}}(t+h) - \vec{\mathbf{f}}(t)}{h}$$



$$z = f(x,\ y)$$

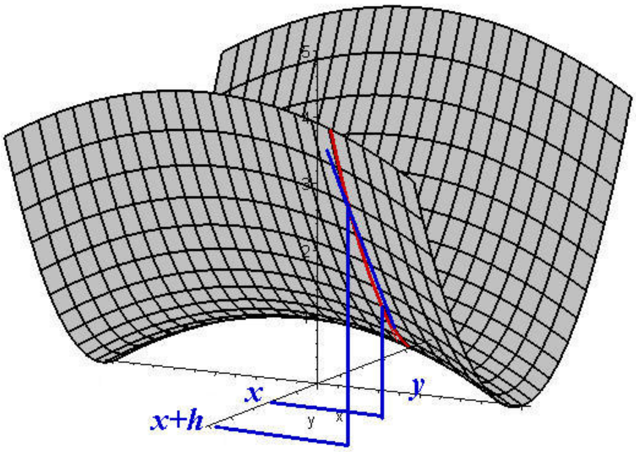


$$f(x,y) = x^2 - \frac{1}{2}y^2 + 1$$



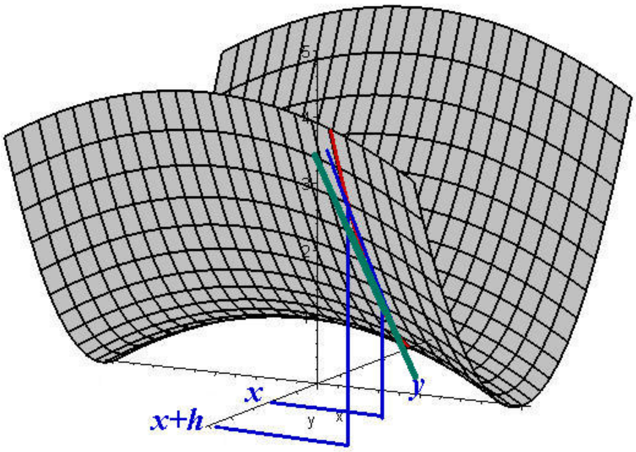
$$f(x,y) = x^2 - \frac{1}{2}y^2 + 1$$

$$\text{Difference Quotient} = \frac{f(x+h,y) - f(x,y)}{h}$$



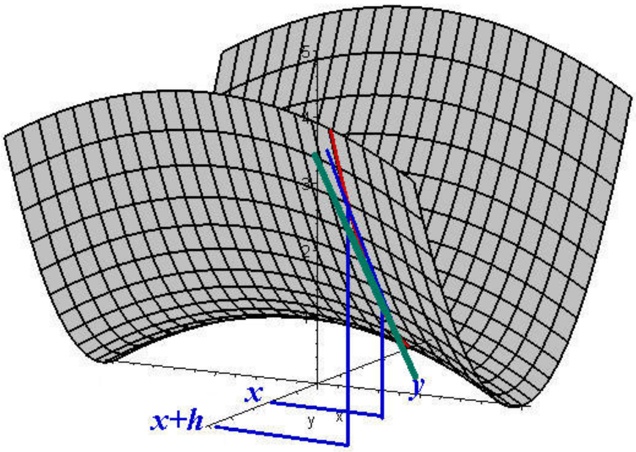
$$f(x,y) = x^2 - \frac{1}{2}y^2 + 1$$

$$\lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$



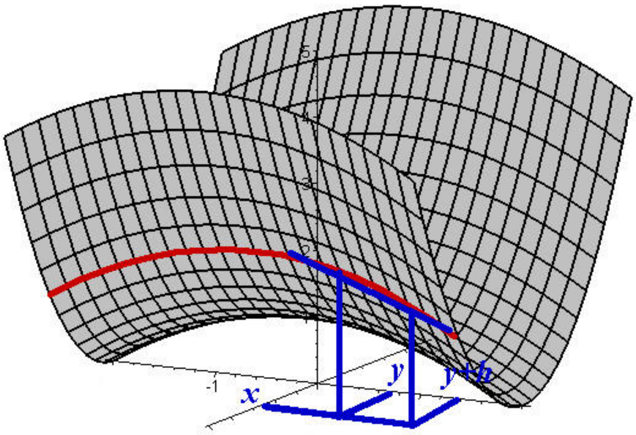
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Other notation: $f_x$

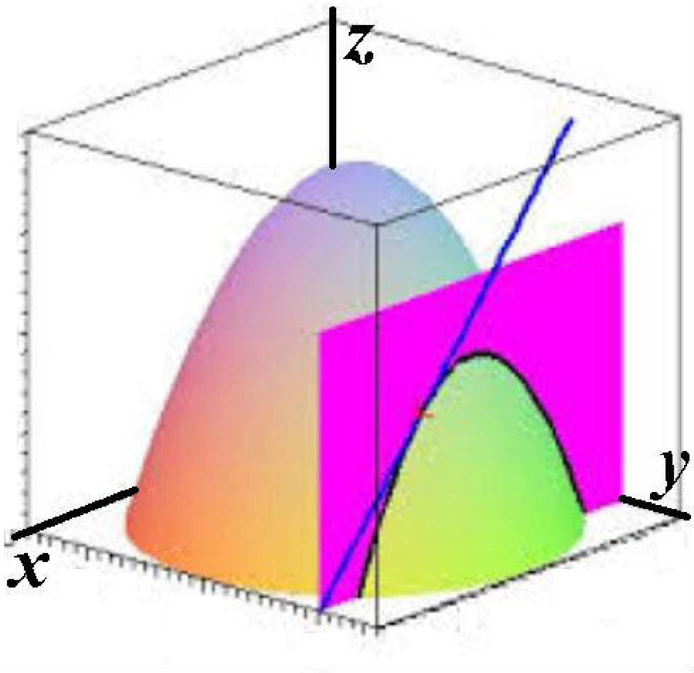


$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

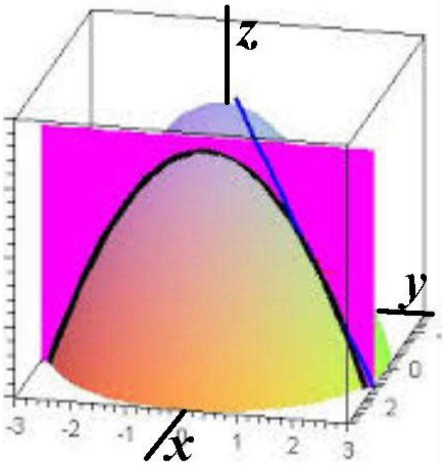
Other natation: $f_y$



$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$



$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$f(x, y) = x^2 - \frac{1}{2}y^2 + 1$$

Find the partial derivative $\frac{\partial f}{\partial x}$

$$f(x,y) = x^2 - \frac{1}{2}y^2 + 1$$

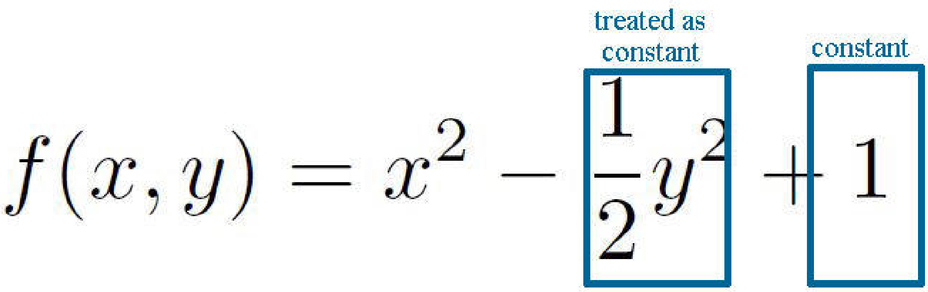
$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{\left[(x+h)^2 - \frac{1}{2}y^2 + 1\right] - \left[x^2 - \frac{1}{2}y^2 + 1\right]}{h}
\end{aligned}$$

$$f(x,y) = x^2 - \frac{1}{2}y^2 + 1$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{\left[(x+h)^2 - \frac{1}{2}y^2 + 1\right] - \left[x^2 - \frac{1}{2}y^2 + 1\right]}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x
\end{aligned}$$

$$f(x, y) = x^2 - \frac{1}{2}y^2 + 1$$

$$\frac{\partial f}{\partial x} = 2x$$



$$f(x,y) = x^2 - \frac{1}{2}y^2 + 1$$

Find the partial derivative $\frac{\partial f}{\partial y}$

$$f(x, y) = x^2 - \frac{1}{2}y^2 + 1$$

$$\frac{\partial f}{\partial y} = -y$$

$$z = x^3y^4 + 2y^2$$

Find the partial derivative of $z$ with respect to $x$

$$z = x^3y^4 + 2y^2$$

Find the partial derivative of $z$ with respect to $x$

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$$\frac{\partial z}{\partial x} = 3x^2y^4$$

$$z = x^3y^4 + 2y^2$$

Find the partial derivative of $z$ with respect to $y$

$$z = x^3y^4 + 2y^2$$

Find the partial derivative of $z$ with respect to $y$

---

$$\frac{\partial z}{\partial y} = 4x^3y^3 + 4y$$

Alternate notation:

$$\frac{\partial}{\partial x}\left(x^3y^4 + 2y^2\right) = 3x^2y^4$$

$$\frac{\partial}{\partial y}\left(x^3y^4 + 2y^2\right) = 4x^3y^3 + 4y$$

$$z = \frac{x^3}{y^2}$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$z = \frac{x^3}{y^2}$$

$$z = \frac{1}{y^2} \cdot x^3$$

$$\frac{\partial z}{\partial x} = \frac{1}{y^2} \cdot 3x^2 = \frac{3x^2}{y^2}$$

$$z = \frac{x^3}{y^2} = x^3 y^{-2}$$

$$\frac{\partial z}{\partial y} = x^3 \cdot \left(-2y^{-3}\right) = \frac{-2x^3}{y^3}$$

Alternate notation:

$$\frac{\partial}{\partial x}\left(\frac{x^3}{y^2}\right) = \frac{3x^2}{y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{x^3}{y^2}\right) = \frac{-2x^3}{y^3}$$

The partial derivative of $\frac{\partial f}{\partial x}$ with respect to $x$ is called the *second partial derivative* of $f$ with respect to $x$.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

Other notation:

$$f_{xx}$$

The partial derivative of $\frac{\partial f}{\partial y}$ with respect to $y$ is called the *second partial derivative* of $f$ with respect to $y$.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Other notation:

$$f_{yy}$$

The partial derivative of $\frac{\partial f}{\partial x}$ with respect to $y$ or the the partial derivative of $\frac{\partial f}{\partial y}$ with respect to $x$ is called a second *mixed* partial derivative.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$z = x^3y^4 + 2y^2$$

Find all four of the second partial derivatives.

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Find all four of the second partial derivatives.

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$$\frac{\partial^2 z}{\partial x^2} \qquad \frac{\partial^2 z}{\partial y^2} \qquad \frac{\partial^2 z}{\partial x \partial y} \qquad \frac{\partial^2 z}{\partial y \partial x}$$

$$z = x^3y^4 + 2y^2$$

Find all four of the second partial derivatives.

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$$\frac{\partial z}{\partial x} = 3x^2y^4 \qquad \frac{\partial z}{\partial y} = 4x^3y^3 + 4y$$

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$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2y^4\right) = 6xy^4$$

$$z = x^3y^4 + 2y^2$$

Find all four of the second partial derivatives.

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$$\frac{\partial z}{\partial x} = 3x^2y^4 \qquad \frac{\partial z}{\partial y} = 4x^3y^3 + 4y$$

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$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(4x^3y^3 + 4y\right) = 12x^3y^2 + 4$$

$$z = x^3y^4 + 2y^2$$

Find all four of the second partial derivatives.

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$$\frac{\partial z}{\partial x} = 3x^2y^4 \qquad \frac{\partial z}{\partial y} = 4x^3y^3 + 4y$$

---

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(4x^3y^3 + 4y\right) = 12x^2y^3$$

$$z = x^3y^4 + 2y^2$$

Find all four of the second partial derivatives.

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$$\frac{\partial z}{\partial x} = 3x^2y^4 \qquad \frac{\partial z}{\partial y} = 4x^3y^3 + 4y$$

---

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(4x^3y^3 + 4y\right) = 12x^2y^3$$

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$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(3x^2y^4\right) = 12x^2y^3$$

$$z = \frac{x^3}{y^2}$$

Find all four of the second partial derivatives.

$$\frac{\partial^2 z}{\partial x^2} \qquad \frac{\partial^2 z}{\partial y^2} \qquad \frac{\partial^2 z}{\partial x \partial y} \qquad \frac{\partial^2 z}{\partial y \partial x}$$

$$z = \frac{x^3}{y^2}$$

$$\frac{\partial z}{\partial x} = \frac{3x^2}{y^2} = 3x^2y^{-2} \qquad \frac{\partial z}{\partial y} = \frac{-2x^3}{y^3} = -2x^3y^{-3}$$

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$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2y^{-2}\right) = 6xy^{-2} = \frac{6x}{y^2}$$

$$z = \frac{x^3}{y^2}$$

$$\frac{\partial z}{\partial x} = \frac{3x^2}{y^2} = 3x^2 y^{-2} \qquad \frac{\partial z}{\partial y} = \frac{-2x^3}{y^3} = -2x^3 y^{-3}$$

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$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-2x^3 y^{-3}\right) = 6x^3 y^{-4} = \frac{6x^3}{y^4}$$

$$z = \frac{x^3}{y^2}$$

$$\frac{\partial z}{\partial x} = \frac{3x^2}{y^2} = 3x^2y^{-2} \qquad \frac{\partial z}{\partial y} = \frac{-2x^3}{y^3} = -2x^3y^{-3}$$

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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-2x^3y^{-3}\right) = -6x^2y^{-3} = -\frac{6x^2}{y^3}$$

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$$\frac{\partial^2 z}{\partial y \partial x} =$$

$$z = \frac{x^3}{y^2}$$

$$\frac{\partial z}{\partial x} = \frac{3x^2}{y^2} = 3x^2y^{-2} \qquad \frac{\partial z}{\partial y} = \frac{-2x^3}{y^3} = -2x^3y^{-3}$$

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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-2x^3y^{-3}\right) = -6x^2y^{-3} = -\frac{6x^2}{y^3}$$

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$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(3x^2y^{-2}\right) = -6x^2y^{-3} = -\frac{6x^2}{y^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

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$$f_{xy} = f_{yx}$$