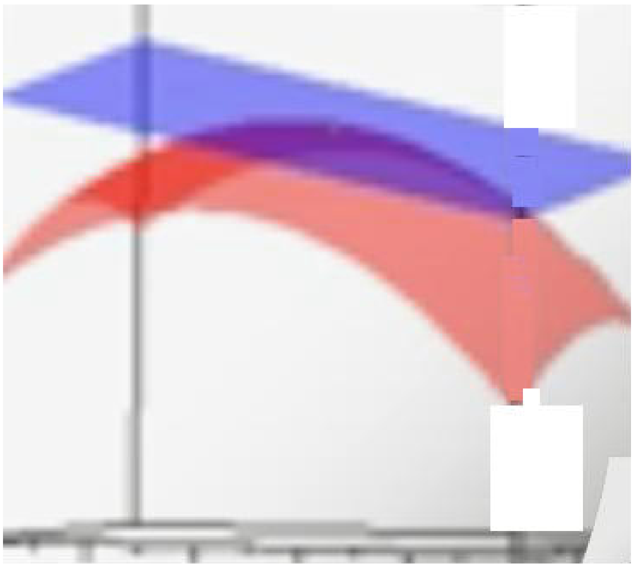
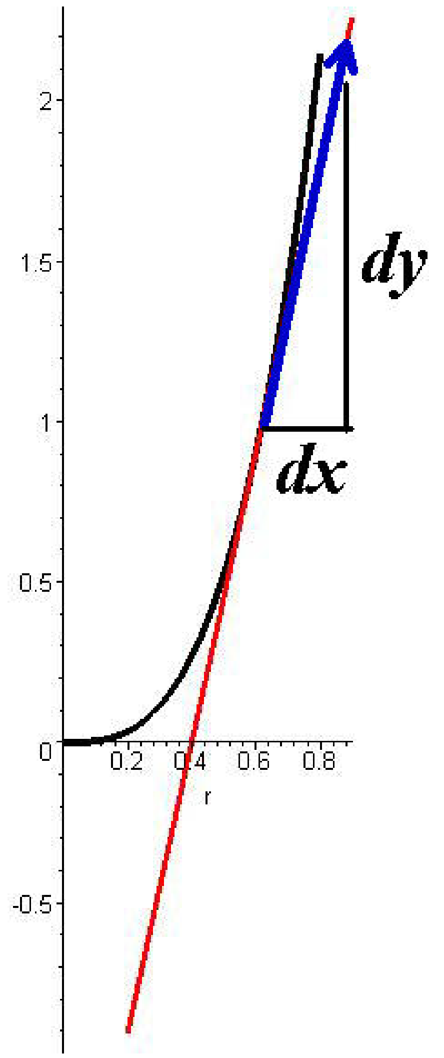
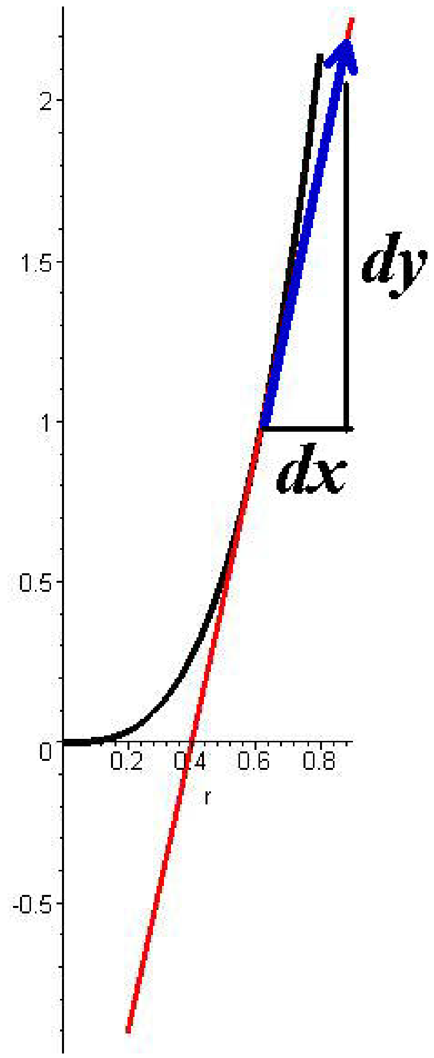
Tangent Planes and Differentials



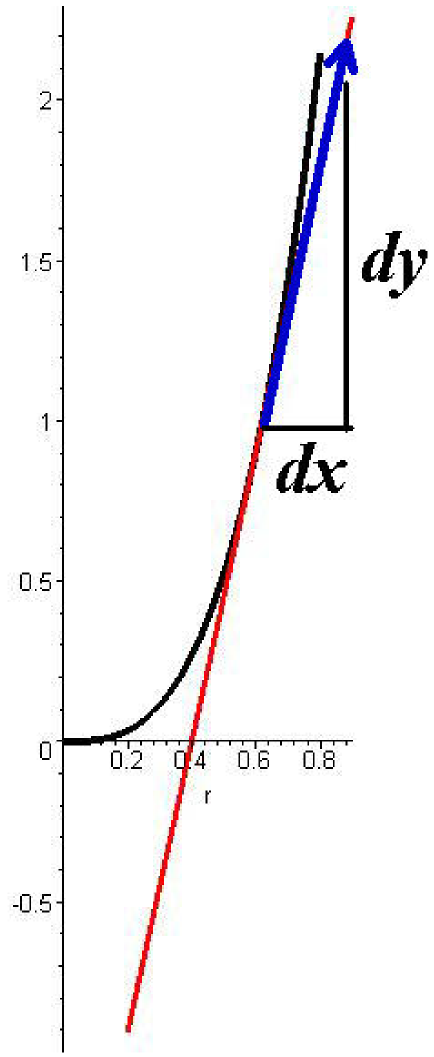
Tangent Vector = $\langle dx,\ dy \rangle$



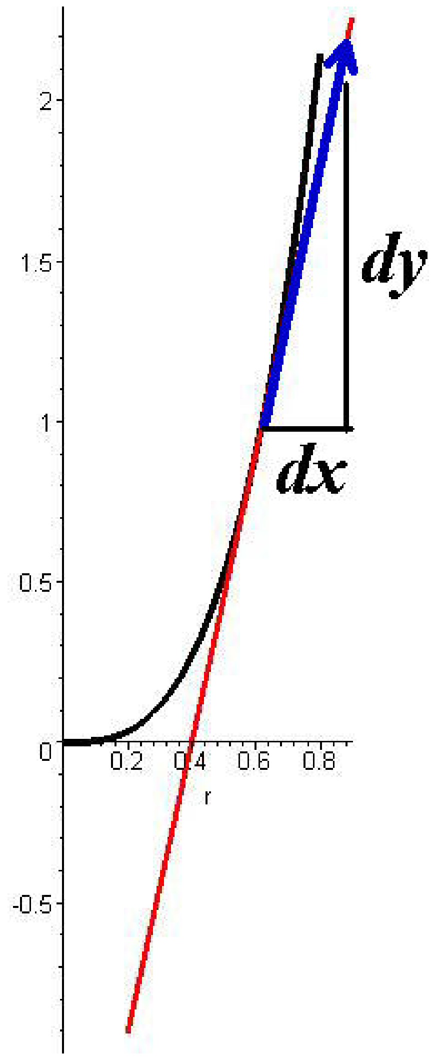
$$\langle dx,\ dy\rangle = \langle dx,\ y'dx\rangle = \langle 1,\ y'\rangle\, dx$$



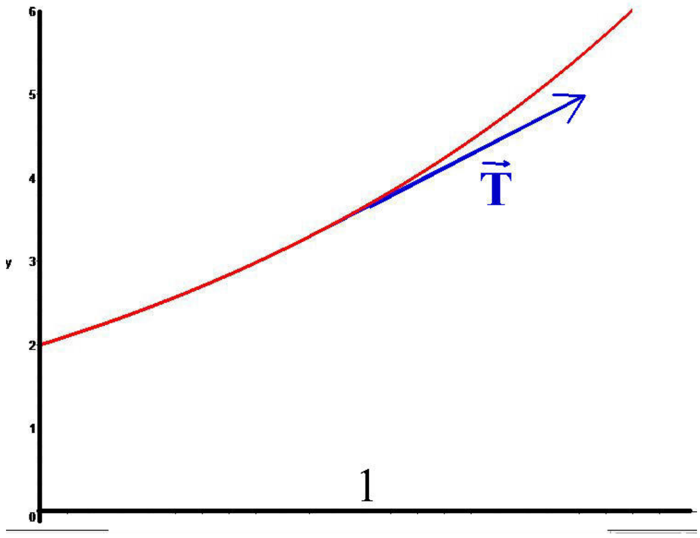
$$\langle dx,\ dy \rangle = \langle dx,\ y'dx \rangle = \langle 1,\ y' \rangle\, dx = (\vec{\mathbf{i}} + y'\vec{\mathbf{j}})dx$$



$$\vec{\mathbf{T}} = \vec{\mathbf{i}} + y'\vec{\mathbf{j}}$$

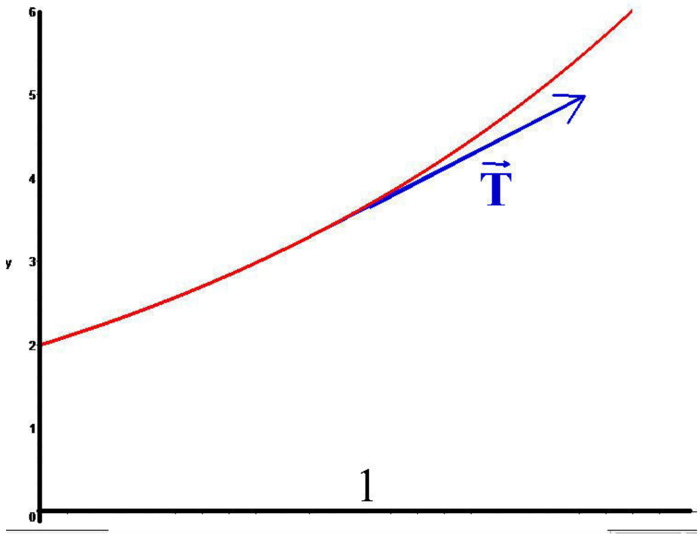


Example: Find $\vec{\mathbf{T}}$ for $y = 2e^{x/2}$, so at $x = 1$



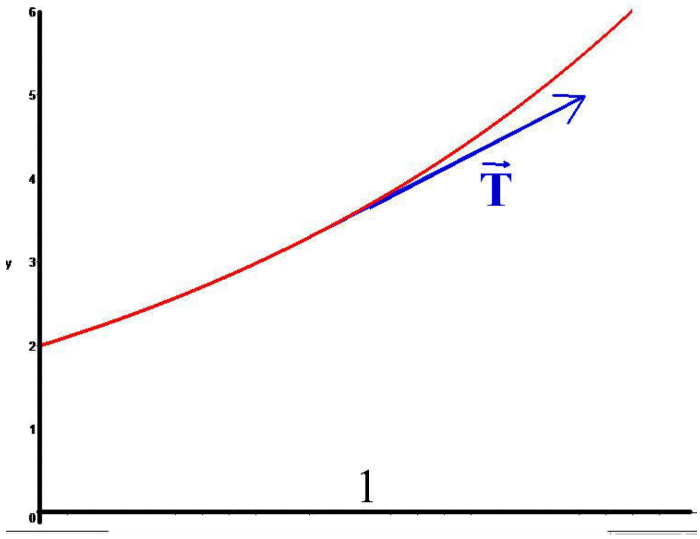
Example: Find $\vec{\mathbf{T}}$ for $y = 2e^{x/2}$, so at $x = 1$

$$y' = e^{x/2} \text{ so at } x = 1,\ y' = e^{1/2}$$



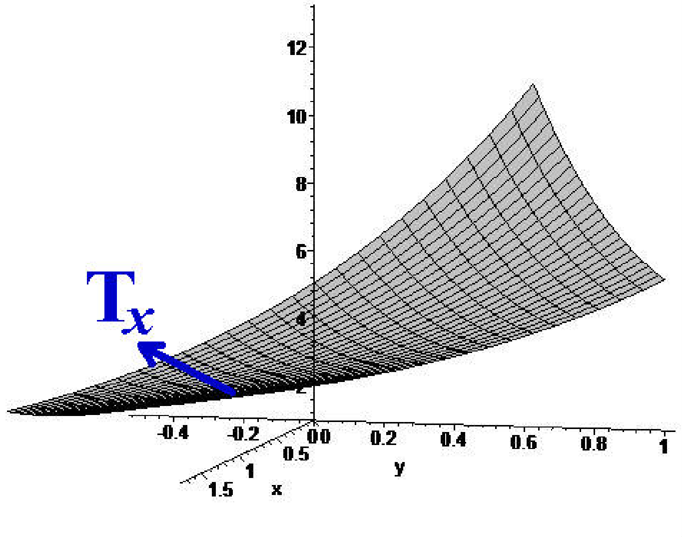
Example: Find $\vec{\mathbf{T}}$ for $y = 2e^{x/2}$, so at $x = 1$

$$\vec{\mathbf{T}} = 1\vec{\mathbf{i}} + y'\vec{\mathbf{j}} = 1\vec{\mathbf{i}} + \sqrt{e}\vec{\mathbf{j}}$$



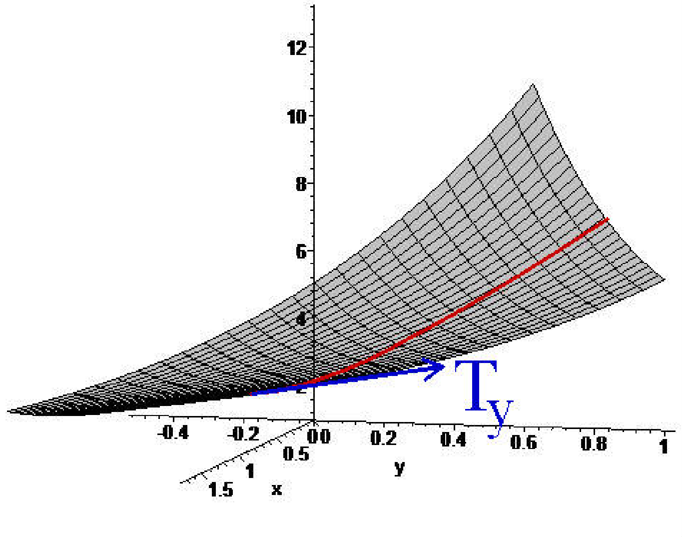
$$z = 2e^{x/2+y}$$

At $y = 0$, $z = 2e^{x/2}$ so $\vec{\mathbf{T}}_x = 1\vec{\mathbf{i}} + \sqrt{e}\vec{\mathbf{k}}$



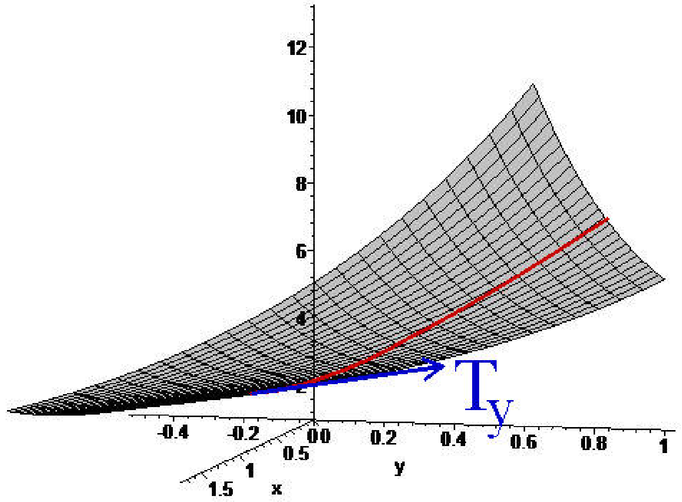
$$z = 2e^{x/2+y}$$

At $x = 1$, $z = 2e^{1/2+y}$



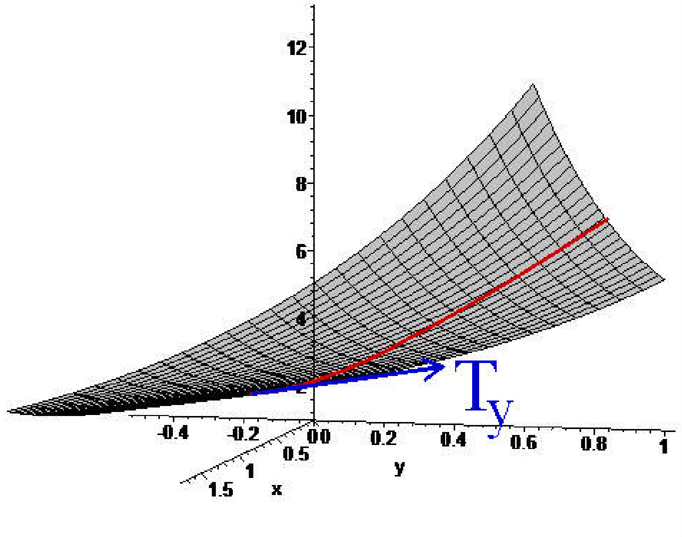
$$z = 2e^{x/2+y}$$

At $x = 1$, $z = 2e^{1/2+y}$ so $\frac{\partial z}{\partial y}(1,\ 0) = 2e^{1/2} = 2\sqrt{e}$



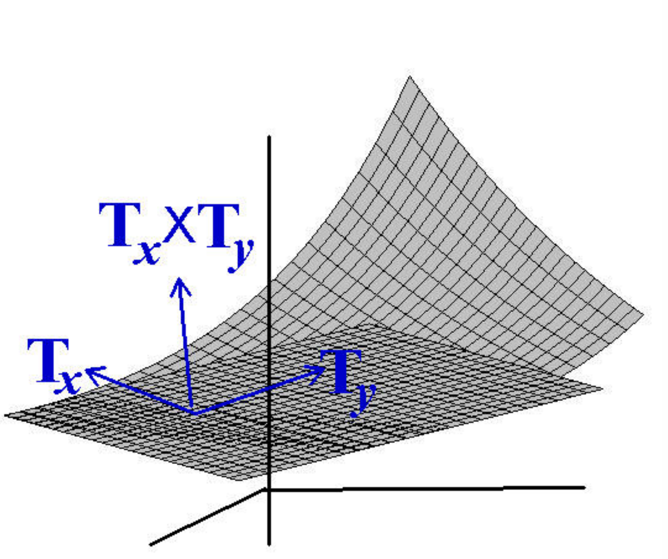
$$z = 2e^{x/2+y}$$

If $\frac{\partial z}{\partial y}(1,\ 0) = 2\sqrt{e}$ then $\vec{\mathbf{T}}_y = 1\vec{\mathbf{j}} + 2\sqrt{e}\,\vec{\mathbf{k}}$



$$z = 2e^{x/2+y}$$

$$\vec{\mathbf{n}} = \vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y$$



$$\vec{\mathbf{n}} = \vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & \sqrt{e} \\ 0 & 1 & 2\sqrt{e} \end{vmatrix} = \langle -\sqrt{e},\ -2\sqrt{e},\ 1 \rangle$$

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$$\vec{\mathbf{n}} \bullet (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$$

$$\langle -\sqrt{e},\ -2\sqrt{e},\ 1 \rangle \bullet (\langle x,\ y,\ z \rangle - \langle 1,\ 0,\ 2\sqrt{e} \rangle) = 0$$

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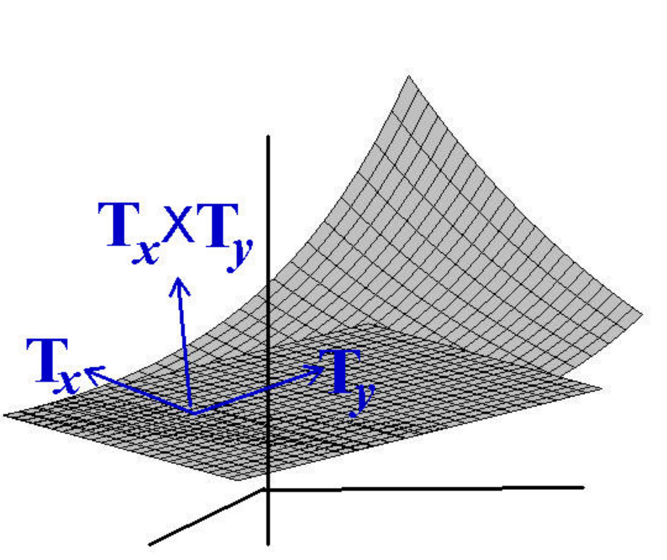
$$z = \sqrt{e}(x + 2y + 1)$$

$$L(x, y) = \sqrt{e}(x + 2y + 1)$$



General Case $\qquad z = f(x, y)$

Find the equation of the plane tangent to this surface at $(x_0,\ y_0,\ z_0)$

$$\vec{\mathbf{n}} \bullet (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$$

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$$\vec{\mathbf{n}} \bullet (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$$

$$\langle -z_x,\ -z_y,\ 1\rangle \bullet \langle x - x_0,\ y - y_0,\ z - z_0\rangle = 0$$

$$-z_x(x - x_0) - z_y(y - y_0) + z - z_0 = 0$$

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$$z = z_0 + z_x(x - x_0) + z_y(y - y_0)$$

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$$z = z_0 + \frac{\partial z}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial z}{\partial y}(x_0, y_0)(y - y_0)$$

Example

Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x+1)^2 - (y+1)^2$ at the point $(-1,\ 0,\ 0)$.

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Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x+1)^2 - (y+1)^2$ at the point $(-1,\ 0,\ 0)$.

$$\frac{\partial z}{\partial x} = -2(x+1) \qquad\qquad \frac{\partial z}{\partial y} = -2(y+1)$$

$$z_x = \frac{\partial z}{\partial x}(-1,\ 0) = 0 \qquad\qquad z_y = \frac{\partial z}{\partial y}(-1,\ 0) = -2$$

Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x+1)^2 - (y+1)^2$ at the point $(-1,\ 0,\ 0)$.

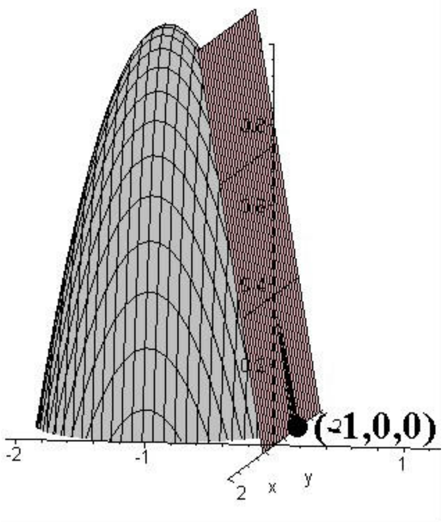
$$z = z_0 + z_x(x - x_0) + z_y(y - y_0)$$

$$z = 0 + (0)(x - (-1)) + (-2)(y - 0)$$

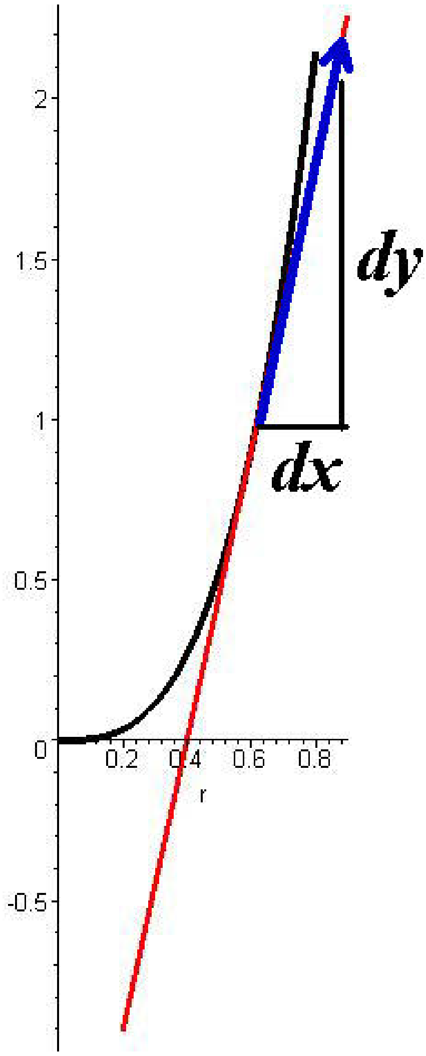
$$z = -2y$$

Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x+1)^2 - (y+1)^2$ at the point $(-1,\ 0,\ 0)$.

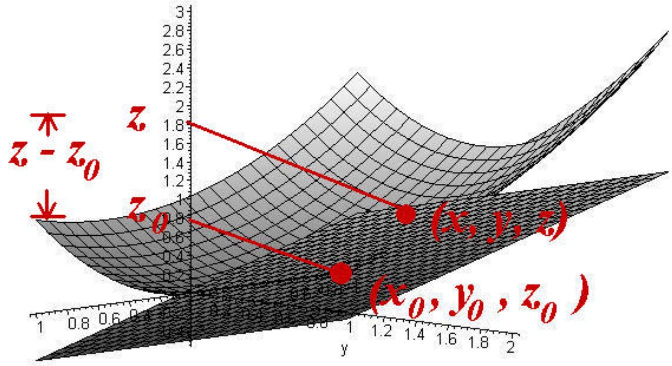
Tangent Plane $\qquad\qquad z = -2y$



In Calculus I, the differential dy is the change in the height of the tangent line.



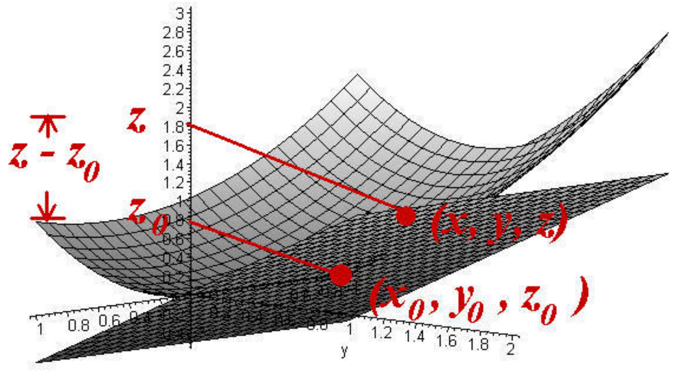
If $z = f(x, y)$, then dz is the change in the height of the tangent plane.



If $z = f(x, y)$, then dz is the change in the height of the tangent plane.

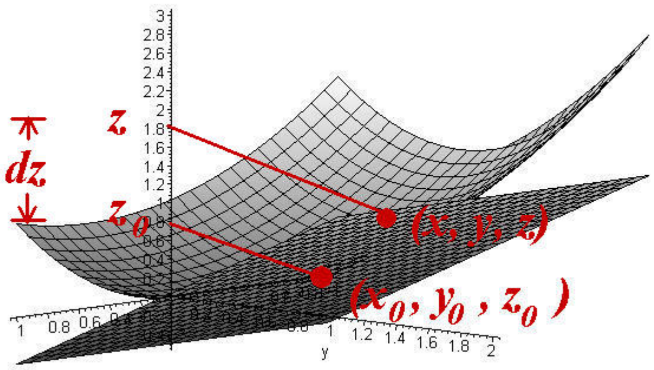
$$z = z_0 + z_x(x - x_0) + z_y(y - y_0)$$

$$z - z_0 = z_x(x - x_0) + z_y(y - y_0)$$

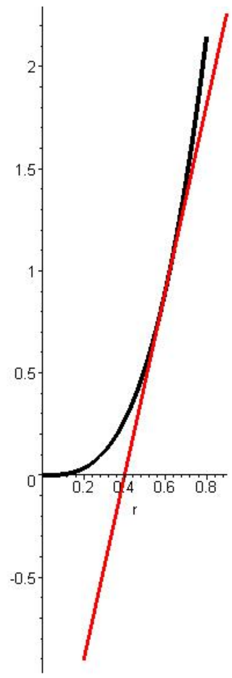


$$z - z_0 = z_x(x - x_0) + z_y(y - y_0)$$

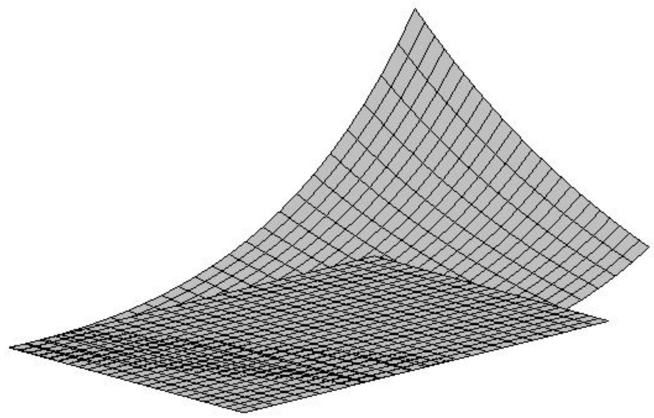
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$



If $y = f(x)$ and dx is small, the differential dy is a good approximation of the change in the height of the curve.



If $z = f(x, y)$ and both dx and dy are small, the differential dz is a good approximation of the change in the height of the surface.



$$z = f(x, y) = 2e^{x/2+y}$$

If we change (x, y) from $(1,\ 0)$ to $(1.02,\ 0.04)$, the change in the height of the surface is:

$$\begin{aligned} f(1.02,\ 0.04) - f(1,\ 0) &= 2e^{1.02/2+0.04} - 2e^{1/2+0} \\ &= 2e^{0.55} - 2e^{0.5} \\ &\approx 0.169 \end{aligned}$$

$$z = f(x, y) = 2e^{x/2+y}$$

$$\frac{\partial z}{\partial x} = e^{x/2+y} \qquad \frac{\partial z}{\partial y} = 2e^{x/2+y}$$

At $x = 1$ and $y = 0$ then:

$$\frac{\partial z}{\partial x} = \sqrt{e} \qquad \frac{\partial z}{\partial y} = 2\sqrt{e}$$

$$z = f(x, y) = 2e^{x/2+y}$$

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At $x = 1$ and $y = 0$ then:

$$\frac{\partial z}{\partial x} = \sqrt{e} \qquad \frac{\partial z}{\partial y} = 2\sqrt{e}$$

(x, y) is changing from $(1,\ 0)$ to $(1.02,\ 0.04)$
so $dx = 0.02$ and $dy = 0.04$

$$
\begin{aligned}
dz &= \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy \\
&= \sqrt{e}(0.02) + 2\sqrt{e}(0.04) \\
&= \sqrt{e}(0.100) \\
&\approx 0.165
\end{aligned}
$$

Compare with $f(1.02,\ 0.04) - f(1,\ 0) = 0.169$

Example: The Ideal Gas Law is $P = \frac{nRT}{V}$.

Suppose V is changed from 1,000 liters to 1,200 liters and the temperature is changed from 100 degrees to 130 degrees.

Use the differential to calculate the relative change in pressure, $\frac{dP}{P}$.

Example: The Ideal Gas Law is $P = \frac{nRT}{V}$.
Suppose V is changed from 1,000 liters to 1,200 liters and the temperature is changed from 100 degrees to 130 degrees.

$$\frac{\partial P}{\partial T} = \frac{nR}{V} \qquad \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$$

$$dP = \frac{\partial P}{\partial T}dT + \frac{\partial P}{\partial V}dV = \frac{nR}{V}dT - \frac{nRT}{V^2}dV$$

$$dP = \frac{\partial P}{\partial T} dT + \frac{\partial P}{\partial V} dV = \frac{nR}{V} dT - \frac{nRT}{V^2} dV$$

$$\frac{dP}{P} = \frac{\frac{nR}{V} dT - \frac{nRT}{V^2} dV}{\frac{nRT}{V}}$$

$$dP = \frac{\partial P}{\partial T} dT + \frac{\partial P}{\partial V} dV = \frac{nR}{V} dT - \frac{nRT}{V^2} dV$$

$$\begin{aligned} \frac{dP}{P} &= \frac{\frac{nR}{V} dT - \frac{nRT}{V^2} dV}{\frac{nRT}{V}} \\ &= \frac{dT}{T} - \frac{dV}{V} \end{aligned}$$

$$\frac{dP}{P} = \frac{dT}{T} - \frac{dV}{V}$$

We are starting at $V = 1,000$ and $T = 100$.
If V is changed from 1,000 liters to 1,200 liters and the temperature is changed from 100 degrees to 130 degrees then $dV = 200$ and $dT = 30$.

$$\frac{dP}{P} = \frac{1}{100}(30) - \frac{200}{1,000} = \frac{1}{10}$$