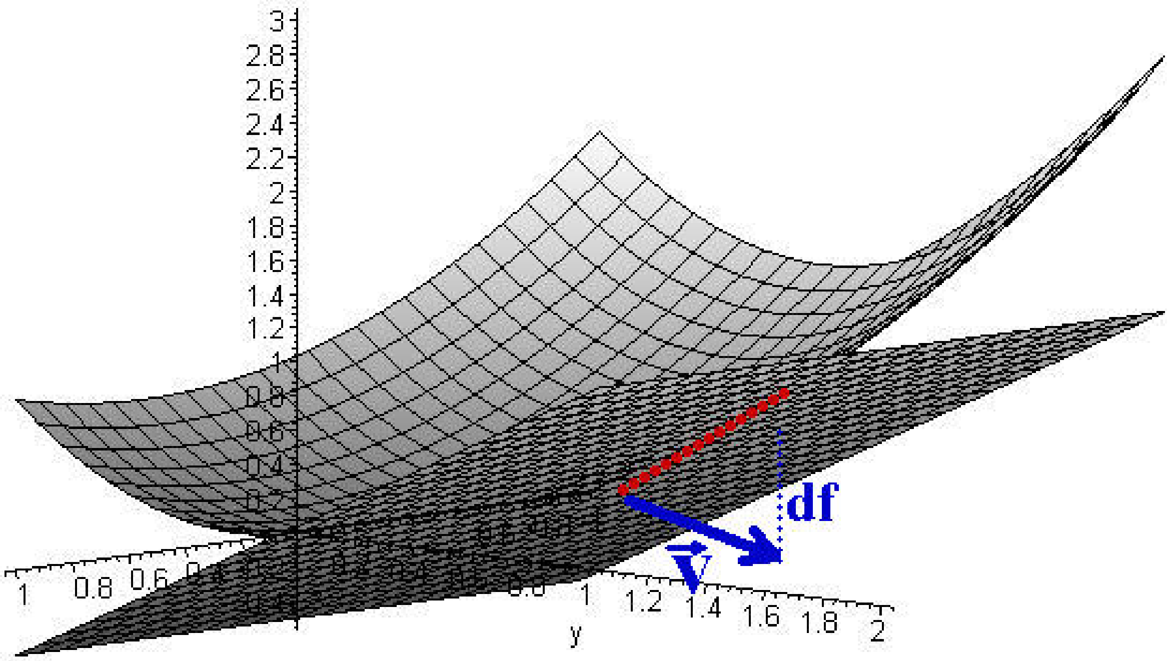
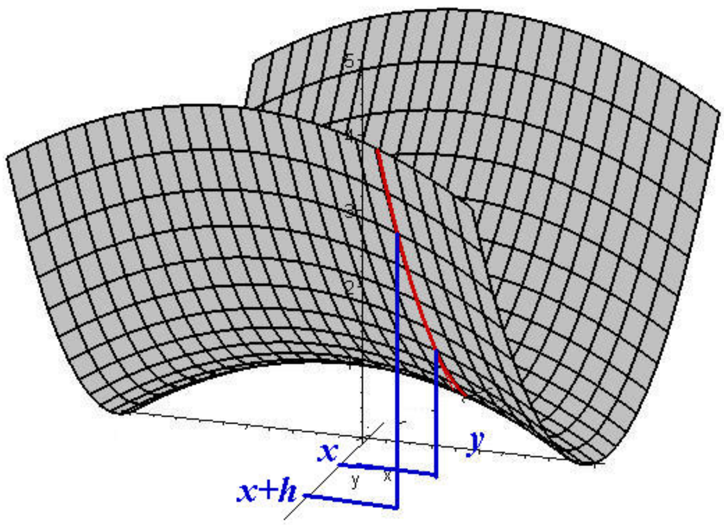
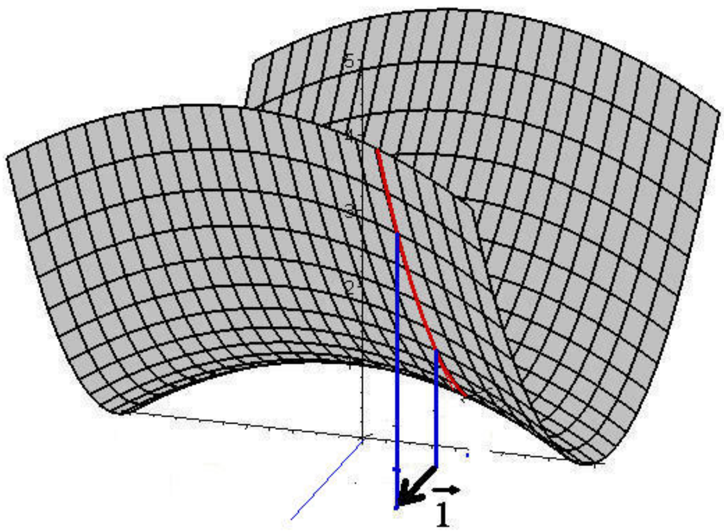
Directional Derivatives



$$\frac{\partial z}{\partial x}$$

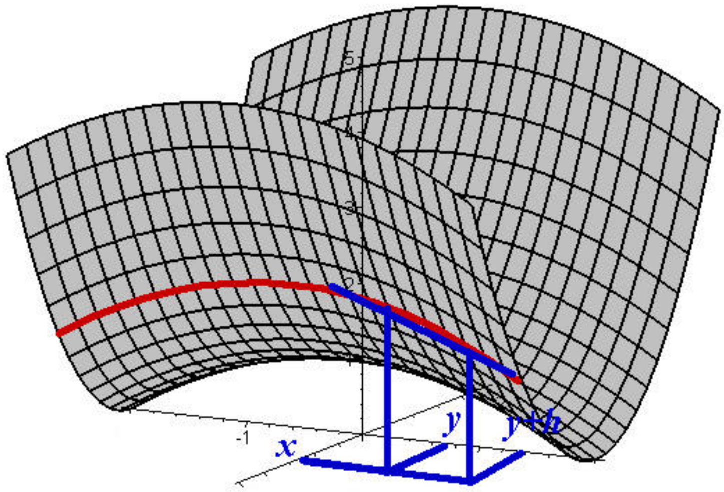


$$\frac{\partial z}{\partial x} = D_{\vec{\mathbf{i}}}\, z$$

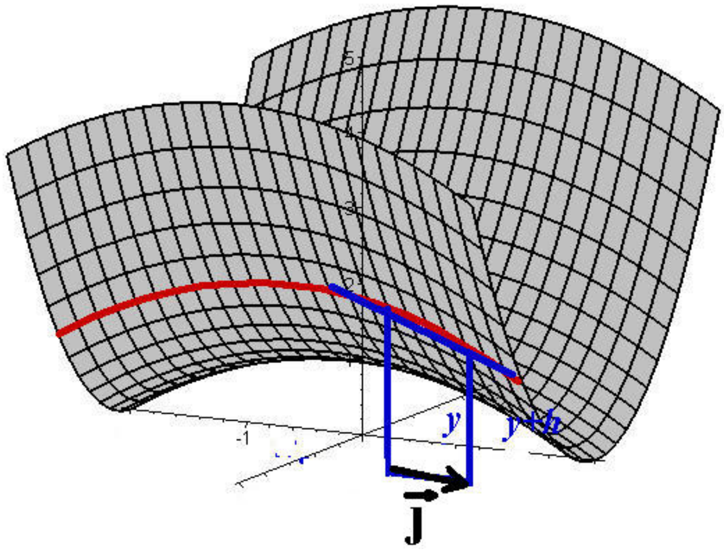


$$D_{\vec{\mathbf{i}}} f(x, y)$$

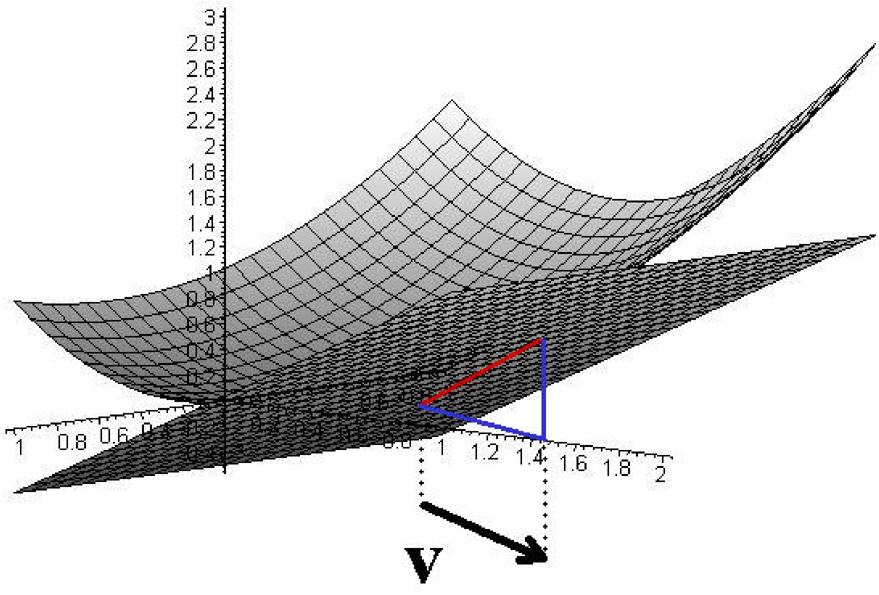
$$\frac{\partial z}{\partial y}$$



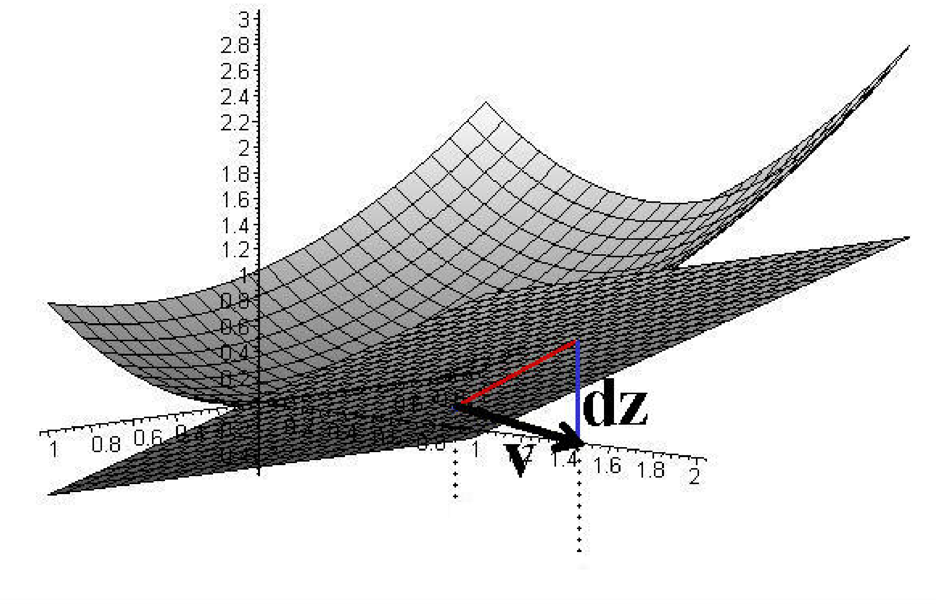
$$\frac{\partial z}{\partial y} = D_{\vec{\mathbf{j}}}\, z$$



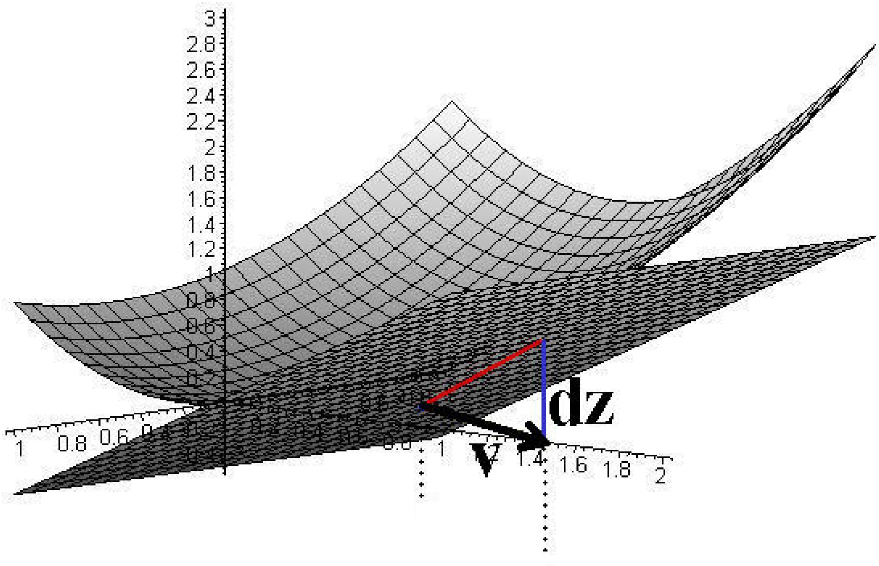
$D_{\vec{\mathbf{v}}}z$



$$D_{\vec{\mathbf{v}}} z = \frac{dz}{|\vec{\mathbf{v}}|}$$

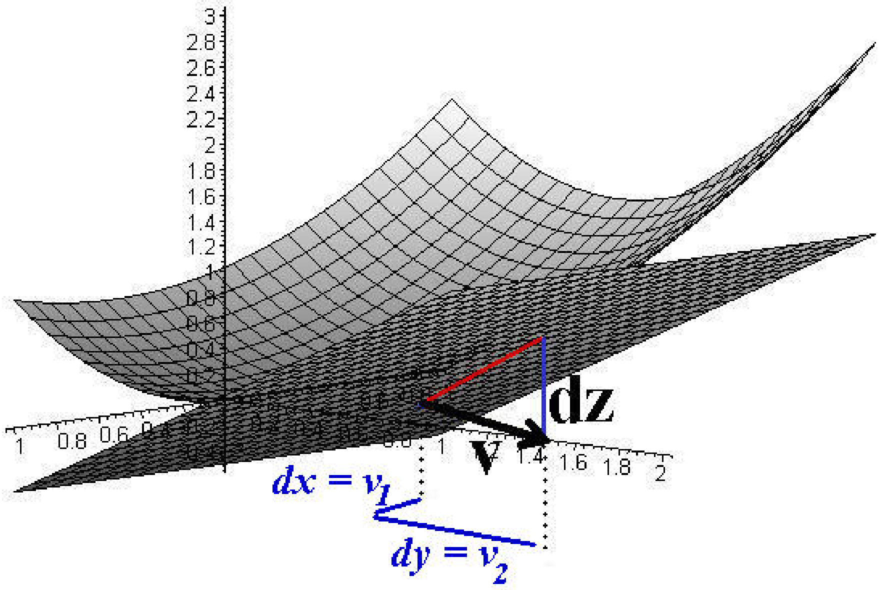


$$D_{\vec{\mathbf{v}}}z = \frac{dz}{|\vec{\mathbf{v}}|} = \frac{\frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}{|\vec{\mathbf{v}}|}$$



$$\vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle \qquad dx = v_1 \qquad dy = v_2$$

$$D_{\vec{\mathbf{v}}}\, z = \frac{dz}{|\vec{\mathbf{v}}|} = \frac{\frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy}{|\vec{\mathbf{v}}|} = \frac{\frac{\partial z}{\partial x}\, v_1 + \frac{\partial z}{\partial y}\, v_2}{|\vec{\mathbf{v}}|}$$



$$D_{\vec{\mathbf{v}}}\, z = \frac{\frac{\partial z}{\partial x}\, v_1 + \frac{\partial z}{\partial y}\, v_2}{|\vec{\mathbf{v}}|} = \frac{\left\langle \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y} \right\rangle \bullet \langle v_1,\ v_2 \rangle}{|\vec{\mathbf{v}}|}$$

This simplifies if $\vec{\mathbf{v}}$ is a unit vector. $|\vec{\mathbf{v}}| = 1$.

$$D_{\vec{\mathbf{v}}}\, z = \left\langle \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y} \right\rangle \bullet \langle v_1,\ v_2 \rangle$$

Definition

The *gradient* of a function $f(x, y)$ is the vector:

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y} \right\rangle$$

$$D_{\vec{\mathbf{v}}}\, z = \frac{\frac{\partial z}{\partial x}\, v_1 + \frac{\partial z}{\partial y}\, v_2}{|\vec{\mathbf{v}}|} = \frac{\left\langle \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y} \right\rangle \bullet \langle v_1,\ v_2 \rangle}{|\vec{\mathbf{v}}|}$$

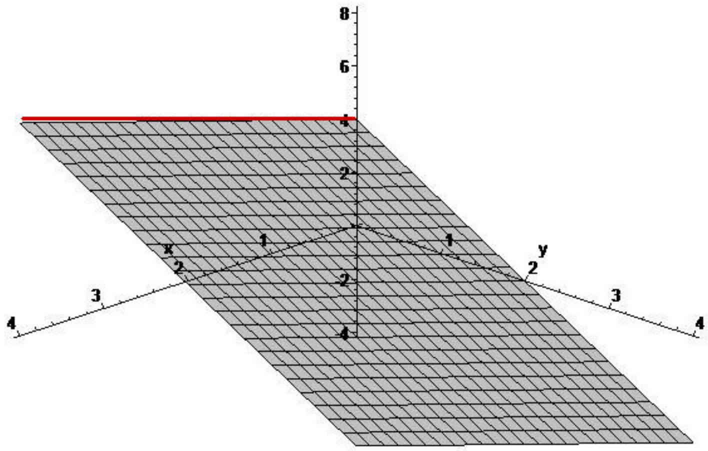
This simplifies if $\vec{\mathbf{v}}$ is a unit vector. $|\vec{\mathbf{v}}| = 1$.

$$D_{\vec{\mathbf{v}}}\, z = \left\langle \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y} \right\rangle \bullet \langle v_1,\ v_2 \rangle$$

$$D_{\vec{\mathbf{v}}}\, z = \nabla z \bullet \vec{\mathbf{v}}$$

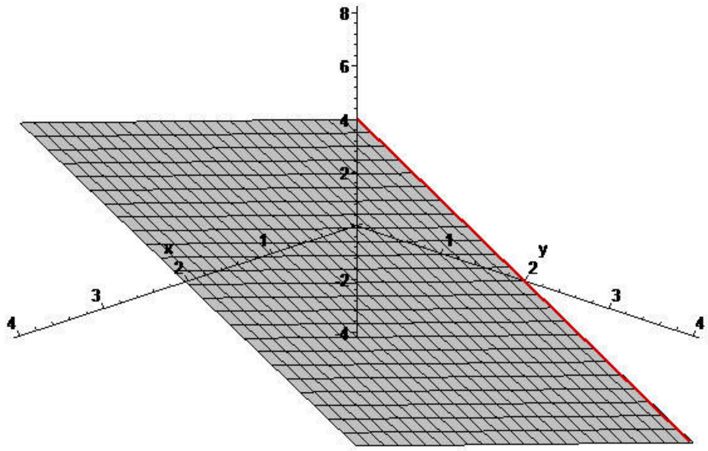
$$z = 4 + x - 2y$$

$$D_{\vec{\mathbf{i}}}\, z = \frac{\partial z}{\partial x} = 1$$



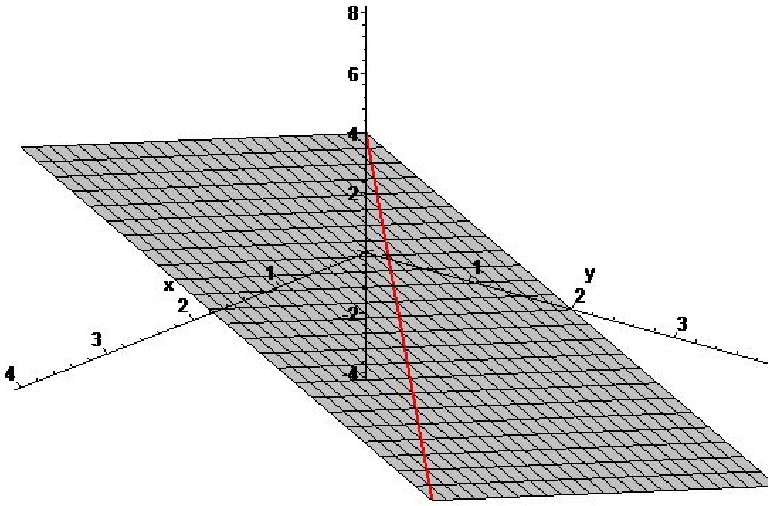
$$z = 4 + x - 2y$$

$$D_{\vec{\mathbf{j}}}\, z = \frac{\partial z}{\partial y} = -2$$



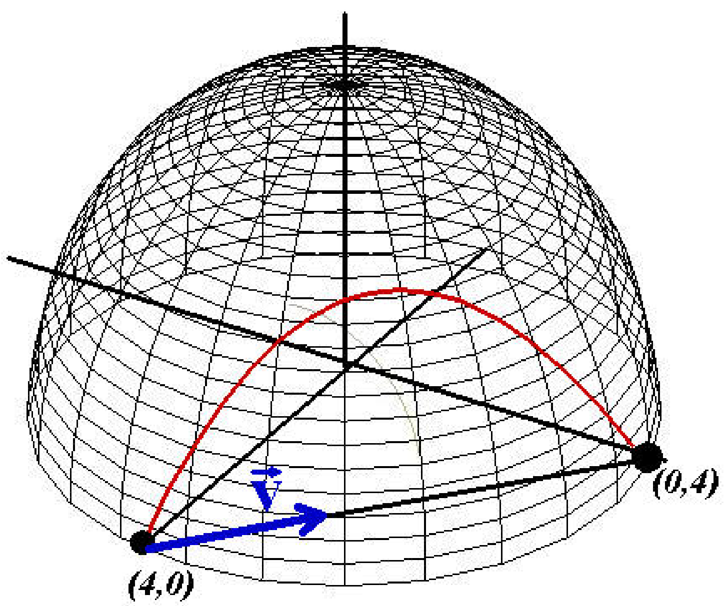
$$z = 4 + x - 2y \qquad \vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle 1,\ 1\rangle$$

$$D_{\vec{\mathbf{v}}}\, z = \nabla z \bullet \vec{\mathbf{v}} = \langle 1,\ -2\rangle \bullet \frac{1}{\sqrt{2}}\langle 1,\ 1\rangle = \frac{-1}{\sqrt{2}}$$



$$f(x, y) = \sqrt{16 - x^2 - y^2}$$

$\vec{\mathbf{v}}$ points in the direction from $(4, 0)$ to $(0, 4)$.
Calculate $D_{\vec{\mathbf{v}}}\, f(x, y)$.



$$f(x,y) = \sqrt{16 - x^2 - y^2}$$

$\vec{\mathbf{v}}$ points in the direction from $(4,0)$ to $(0,4)$.
Calculate $D_{\vec{\mathbf{v}}}\, f(x,y)$.

$$D_{\vec{\mathbf{v}}}\, f(x,y) = \nabla f \bullet \vec{\mathbf{v}}$$

$\vec{\mathbf{v}}$ is in the direction $\vec{\mathbf{u}} = \langle 0,\ 4\rangle - \langle 4,\ 0\rangle = 4\langle -1,\ 1\rangle$

$$\vec{\mathbf{v}} = \frac{1}{|\vec{\mathbf{u}}|}\vec{\mathbf{u}} = \frac{1}{\sqrt{2}}\langle 1,\ -1\rangle$$

$$f(x,y) = \sqrt{16 - x^2 - y^2}$$

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{16 - x^2 - y^2}} \qquad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{16 - x^2 - y^2}}$$

$$\nabla f(x,y) = \left\langle \frac{-x}{\sqrt{16 - x^2 - y^2}}, \ \frac{-y}{\sqrt{16 - x^2 - y^2}} \right\rangle$$

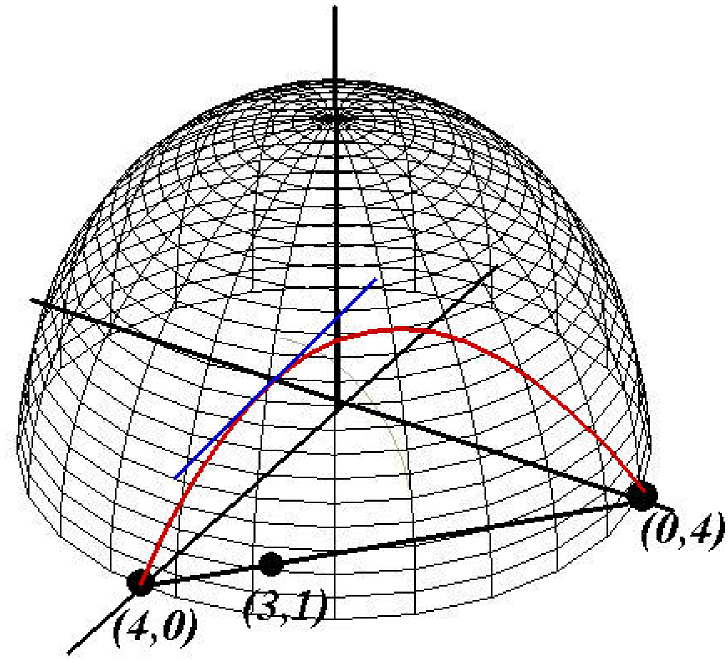
$$\vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle -1,\ 1\rangle$$

$$\nabla f(x, y) = \left\langle \frac{-x}{\sqrt{16 - x^2 - y^2}},\ \frac{-y}{\sqrt{16 - x^2 - y^2}} \right\rangle$$

$$\nabla f(3, 1) = \left\langle \frac{-3}{\sqrt{6}},\ \frac{-1}{\sqrt{6}} \right\rangle$$

$$D_{\vec{\mathbf{v}}} f(3, 1) = \left\langle \frac{-3}{\sqrt{6}},\ \frac{-1}{\sqrt{6}} \right\rangle \bullet \frac{1}{\sqrt{2}}\langle -1,\ 1\rangle = \frac{1}{\sqrt{3}}$$

$$D_{\vec{\mathbf{v}}} f(3,1) = \frac{1}{\sqrt{3}}$$



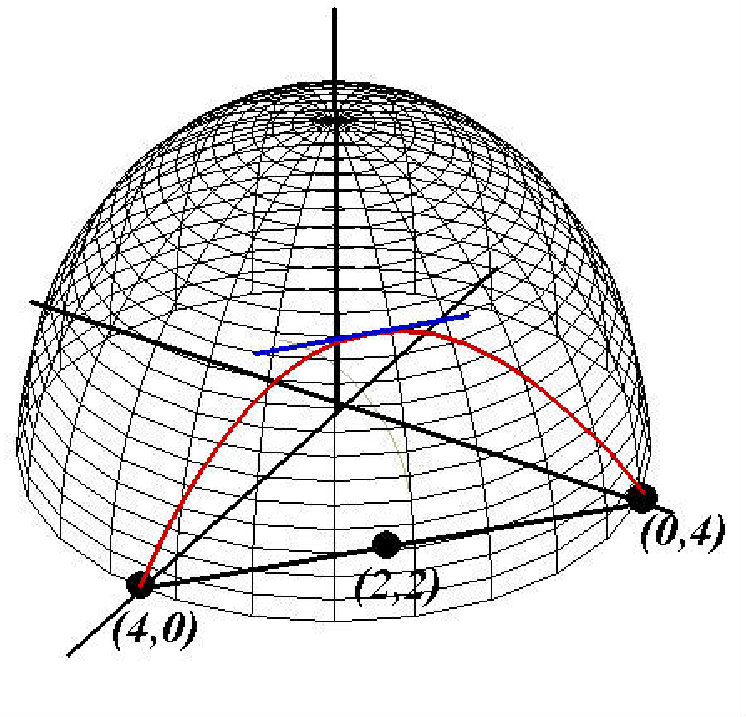
$$\vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle -1,\ 1\rangle$$

$$\nabla f(x,y) = \left\langle \frac{-x}{\sqrt{16-x^2-y^2}},\ \frac{-y}{\sqrt{16-x^2-y^2}} \right\rangle$$

$$\nabla f(2,2) = \left\langle \frac{-2}{\sqrt{8}},\ \frac{-2}{\sqrt{8}} \right\rangle$$

$$D_{\vec{\mathbf{v}}} f(2,2) = \left\langle \frac{-2}{\sqrt{8}},\ \frac{-2}{\sqrt{8}} \right\rangle \bullet \frac{1}{\sqrt{2}}\langle -1,\ 1\rangle = 0$$

$$D_{\vec{\mathbf{v}}} f(2,2) = 0$$



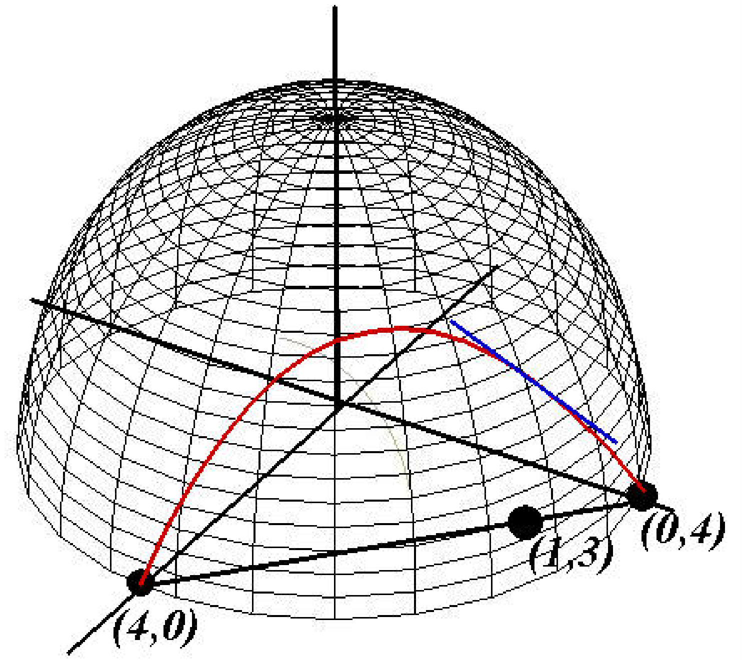
$$\vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle -1,\ 1\rangle$$

$$\nabla f(x,y) = \left\langle \frac{-x}{\sqrt{16-x^2-y^2}},\ \frac{-y}{\sqrt{16-x^2-y^2}} \right\rangle$$

$$\nabla f(1,3) = \left\langle \frac{-1}{\sqrt{6}},\ \frac{-3}{\sqrt{6}} \right\rangle$$

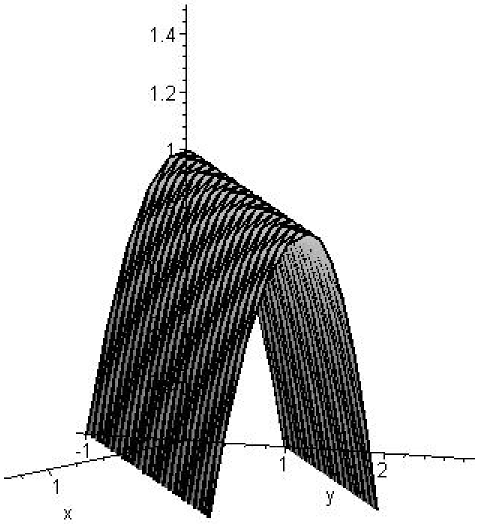
$$D_{\vec{\mathbf{v}}} f(1,3) = \left\langle \frac{-1}{\sqrt{6}},\ \frac{-3}{\sqrt{6}} \right\rangle \bullet \frac{1}{\sqrt{2}}\langle -1,\ 1\rangle = -\frac{1}{\sqrt{3}}$$

$$D_{\vec{\mathbf{v}}} f(1,3) = -\frac{1}{\sqrt{3}}$$



$$f(x,y) = 1 - (y - 2x)^2 \qquad \vec{\mathbf{v}} = \frac{1}{\sqrt{5}}\langle 1,\ 2\rangle$$

Calculate $D_{\vec{\mathbf{v}}}\, f(x,y)$



$$f(x,y) = 1 - (y-2x)^2 \qquad \vec{\mathbf{v}} = \frac{1}{\sqrt{5}}\langle 1,\ 2\rangle$$

$$\frac{\partial f}{\partial x} = 4(y-2x) \qquad \frac{\partial f}{\partial y} = -2(y-2x)$$

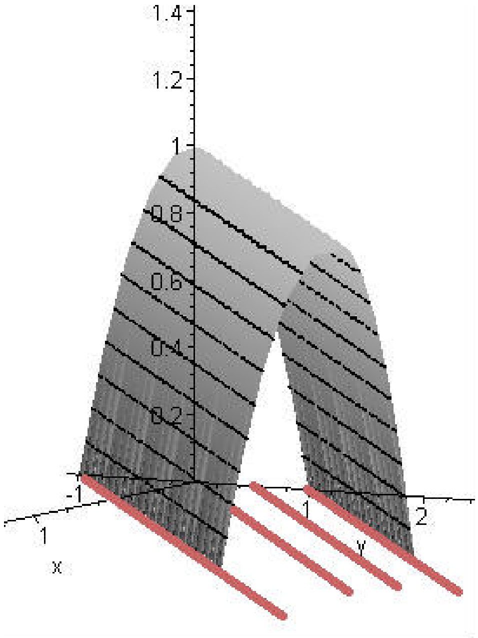
$$\nabla f = \langle 4(y-2x),\ -2(y-2x)\rangle$$

$$\nabla f = \langle 4(y-2x),\ -2(y-2x) \rangle$$

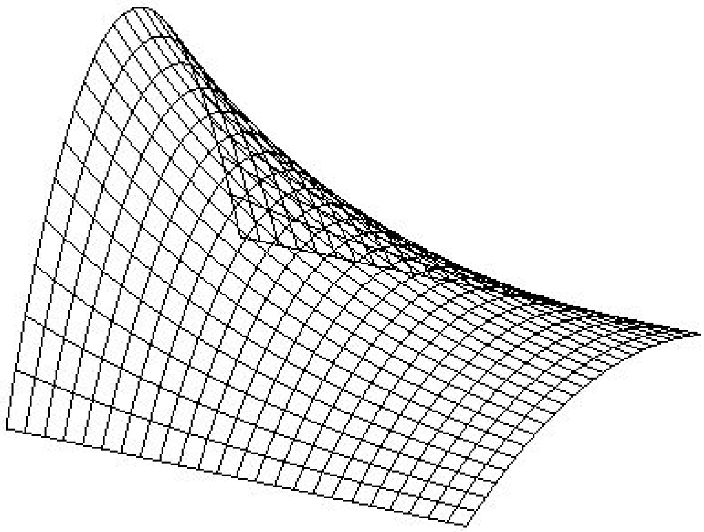
$$\begin{aligned} D_{\vec{\mathbf{v}}}\, f &= \nabla f \bullet \vec{\mathbf{v}} \\ &= \langle 4(y-2x),\ -2(y-2x) \rangle \bullet \frac{1}{\sqrt{5}} \langle 1,\ 2 \rangle \\ &= 0 \end{aligned}$$

$$f(x,y) = 1 - (y-2x)^2 \qquad \vec{\mathbf{v}} = \frac{1}{\sqrt{5}}\langle 1,\ 2\rangle$$

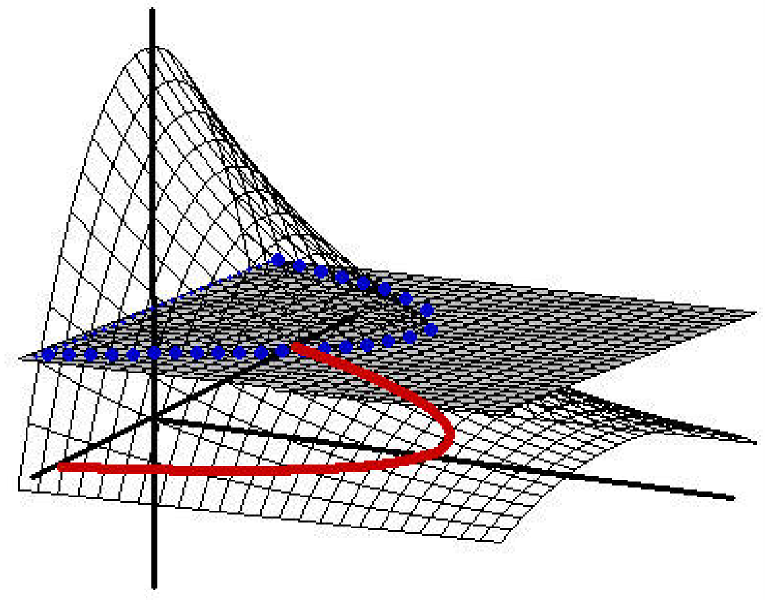
$$D_{\vec{\mathbf{v}}}\, f(x,y) = 0$$



$$z = f(x, y)$$

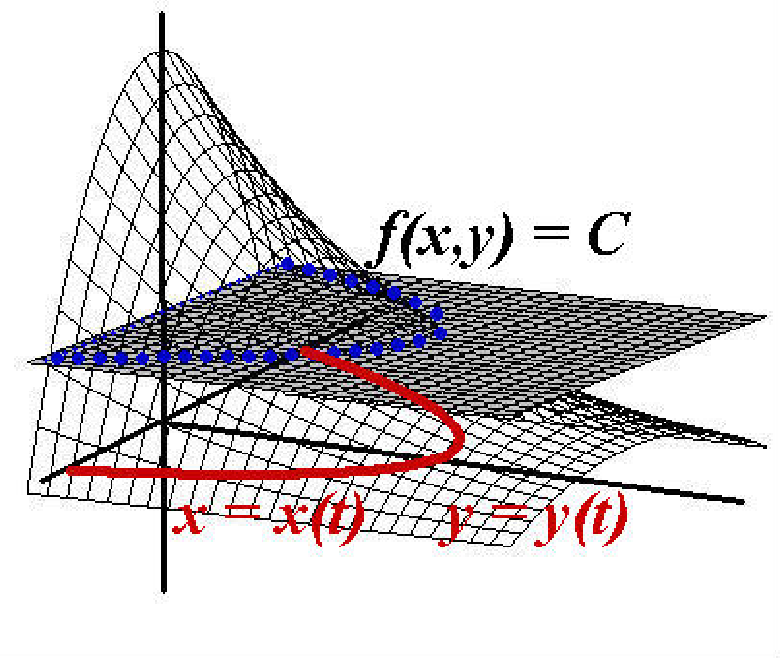


A curve in the xy plane is a level set (also called a contour) if $f(x, y) = C$ for all points on this curve.

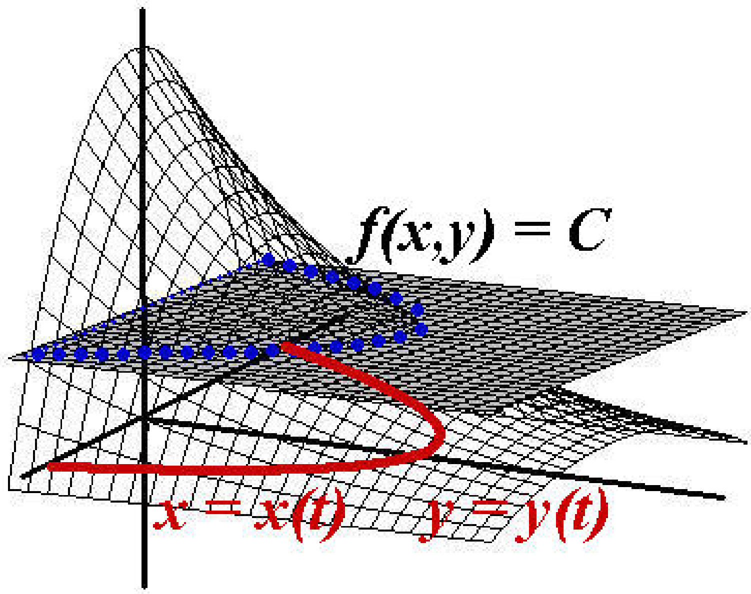


Suppose the equation of the contour is given by the parametric equations:

$$x = x(t) \qquad y = y(t)$$



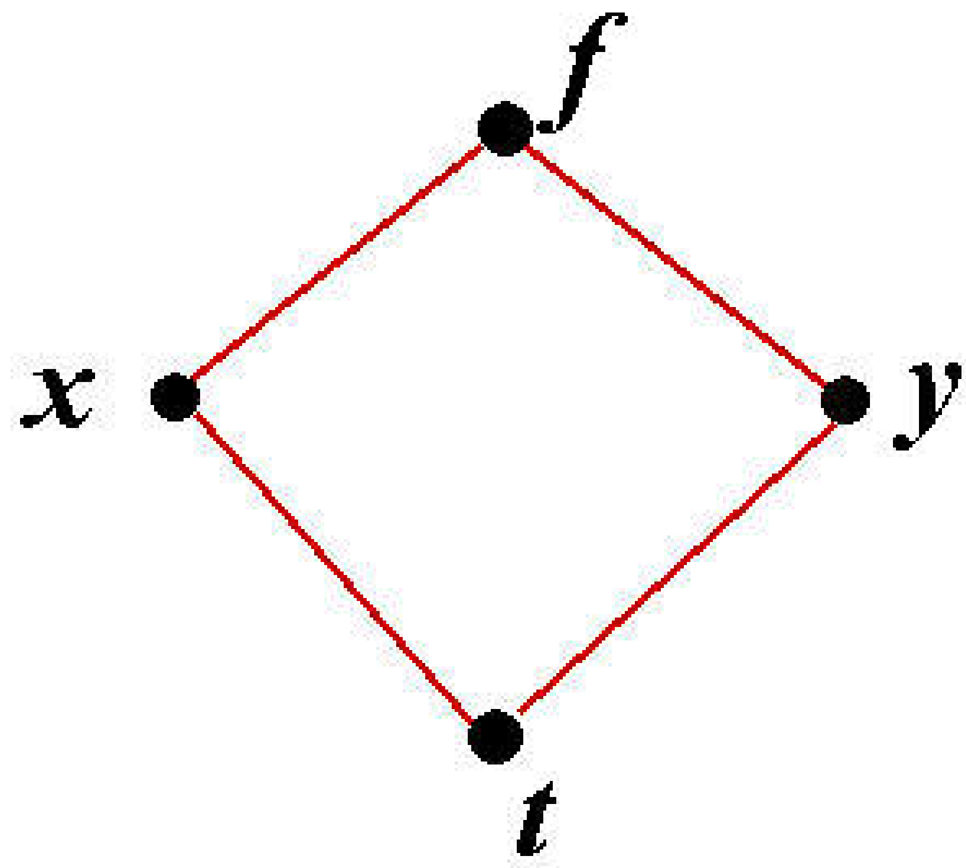
$$f(x(t),\ y(t)) = C$$



$$f(x(t),\ y(t)) = C$$

$$\frac{d}{dt}(f(x(t),\ y(t)) = \frac{d}{dt}(C)$$

$$\frac{d}{dt}(f(x(t),\ y(t)) = 0$$


f
x
y
t

$$f(x(t),\ y(t)) = C$$

$$\frac{d}{dt}(f(x(t),\ y(t)) = 0$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y} \right\rangle \bullet \left\langle \frac{dx}{dt},\ \frac{dy}{dt} \right\rangle = 0$$

$$f(x(t),\ y(t)) = C$$

$$\frac{d}{dt}(f(x(t),\ y(t)) = 0$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y} \right\rangle \bullet \left\langle \frac{dx}{dt},\ \frac{dy}{dt} \right\rangle = 0$$

$$\nabla f \bullet \vec{\mathbf{r}}'(t) = 0$$

$$\nabla f \bullet \vec{\mathbf{r}}'(t) = 0$$

$$\nabla f \bullet \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \frac{0}{|\vec{\mathbf{r}}'(t)|}$$

$$\nabla f \bullet \vec{\mathbf{T}} = 0$$

$$D_{\vec{\mathbf{T}}}\, f = 0$$

$$\nabla f \bullet \vec{\mathbf{r}}'(t) = 0$$

$$\nabla f \bullet \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \frac{0}{|\vec{\mathbf{r}}'(t)|}$$

$$\nabla f \bullet \vec{\mathbf{T}} = 0$$

$$D_{\vec{\mathbf{T}}}\, f = 0$$

Conclusion: $\vec{\mathbf{T}}$ is a unit vector tangent to a level curve of f then $D_{\vec{\mathbf{T}}}\, f = 0$

$$\nabla f \bullet \vec{\mathbf{r}}'(t) = 0$$

$$\nabla f \bullet \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \frac{0}{|\vec{\mathbf{r}}'(t)|}$$

$$\nabla f \bullet \vec{\mathbf{T}} = 0$$

$$D_{\vec{\mathbf{T}}}\, f = 0$$

Conclusion: $\vec{\mathbf{T}}$ is a unit vector tangent to a level curve of f then $\vec{\mathbf{T}}$ is perpendicular to ∇f