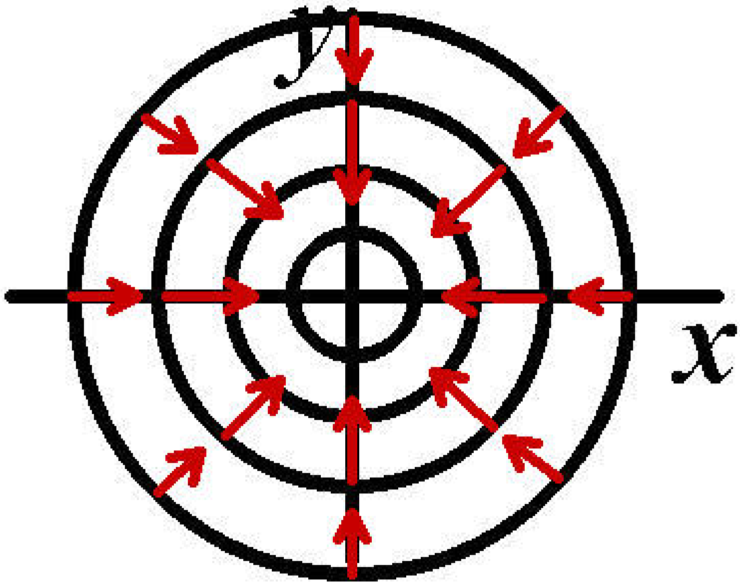
Gradients



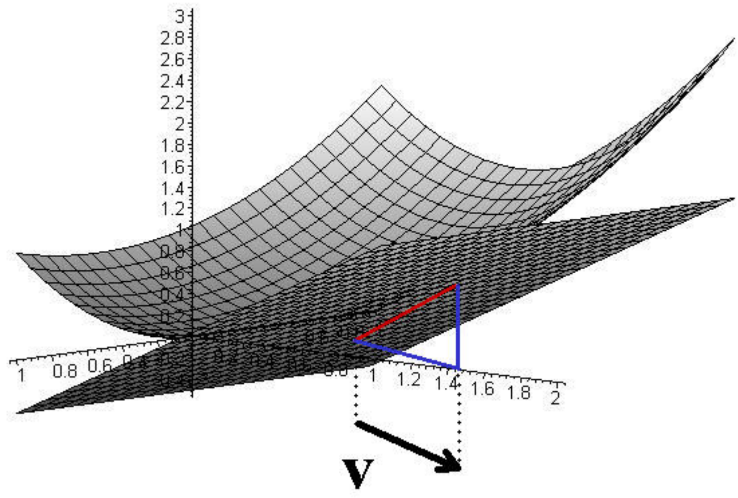
$$f = f(x, y)$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y} \right\rangle$$

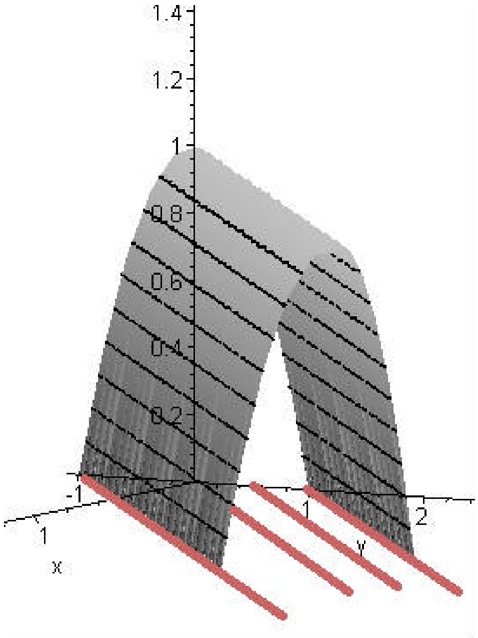
$$f = f(x, y, z)$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \; \frac{\partial f}{\partial y}, \; \frac{\partial f}{\partial z} \right\rangle$$

$$D_{\vec{\mathbf{v}}} f = \nabla f \bullet \vec{\mathbf{v}}$$

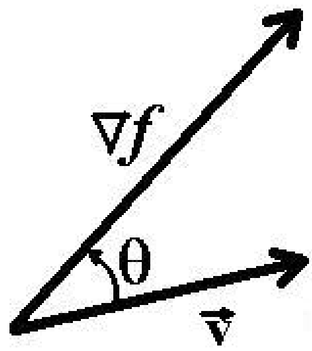


If $\vec{\mathbf{v}}$ is along a level curve then $D_{\vec{\mathbf{v}}}f = 0$



In what direction do we get the biggest value of the directional derivative?

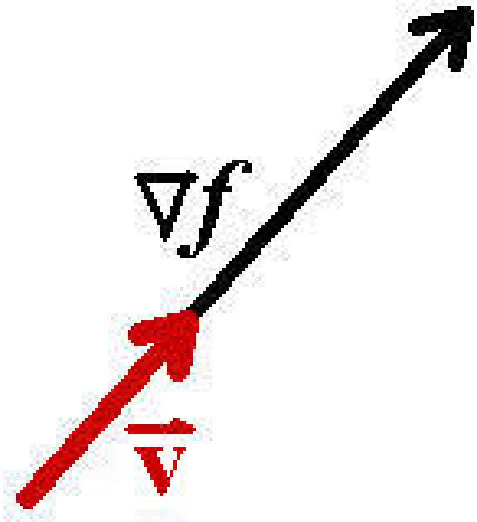
$$\begin{aligned} D_{\vec{\mathbf{v}}} f &= \nabla f \bullet \vec{\mathbf{v}} \\ &= |\nabla f||\vec{\mathbf{v}}| \cos\theta \end{aligned}$$



In what direction do we get the biggest value of the directional derivative?

$$\begin{aligned} D_{\vec{\mathbf{v}}} f &= \nabla f \bullet \vec{\mathbf{v}} \\ &= |\nabla f||\vec{\mathbf{v}}| \cos\theta \\ &= |\nabla f| \cos\theta \end{aligned}$$

The biggest this gets is when $\cos\theta = 1$.
This happens with $\theta = 0$



∇f
$\vec{v}$

We get the largest possible value of $D_{\vec{\mathbf{v}}} f$ when $\vec{\mathbf{v}}$ points in the same direction as ∇f.

So in what direction is ∇f?

Along a level curve, $\vec{\mathbf{r}} = \langle x(t),\ y(t)\rangle$, $f(x, y)$ stays constant.

$$f(x(t), y(t)) = C$$

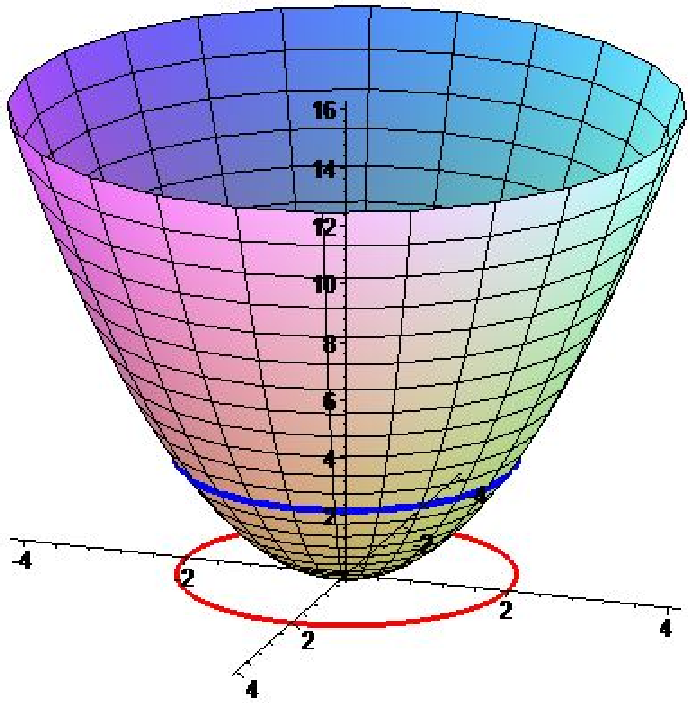
$$\frac{d}{dt}\left(f(x(t), y(t))\right) = 0$$

$$\nabla f \bullet \vec{\mathbf{r}}'(t) = 0$$

Conclusion: $\frac{d\vec{\mathbf{r}}}{dt}$ is perpendicular to ∇f

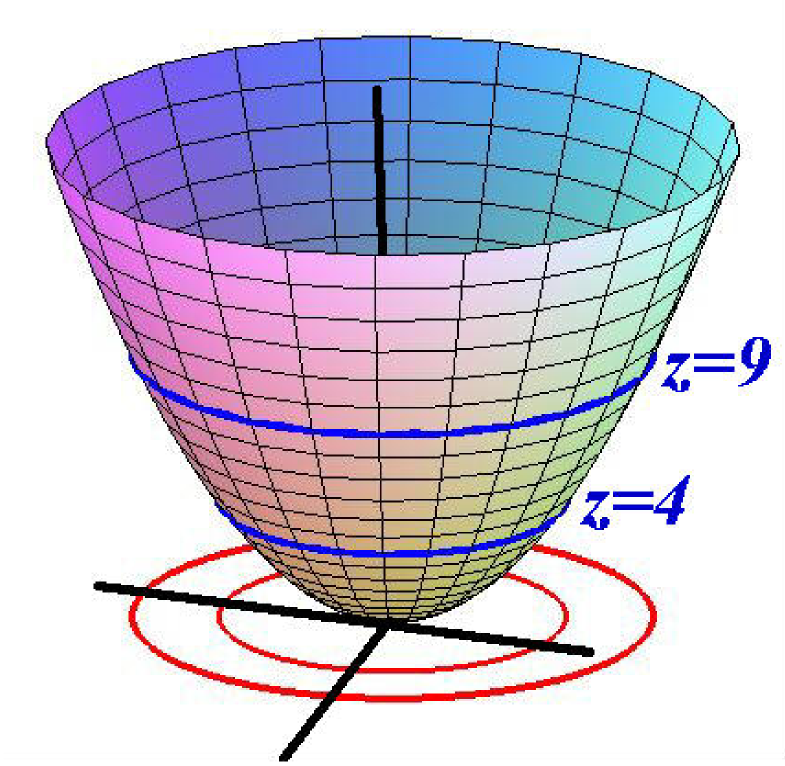
$$z = x^2 + y^2$$

The red curve in the xy plane is a level set

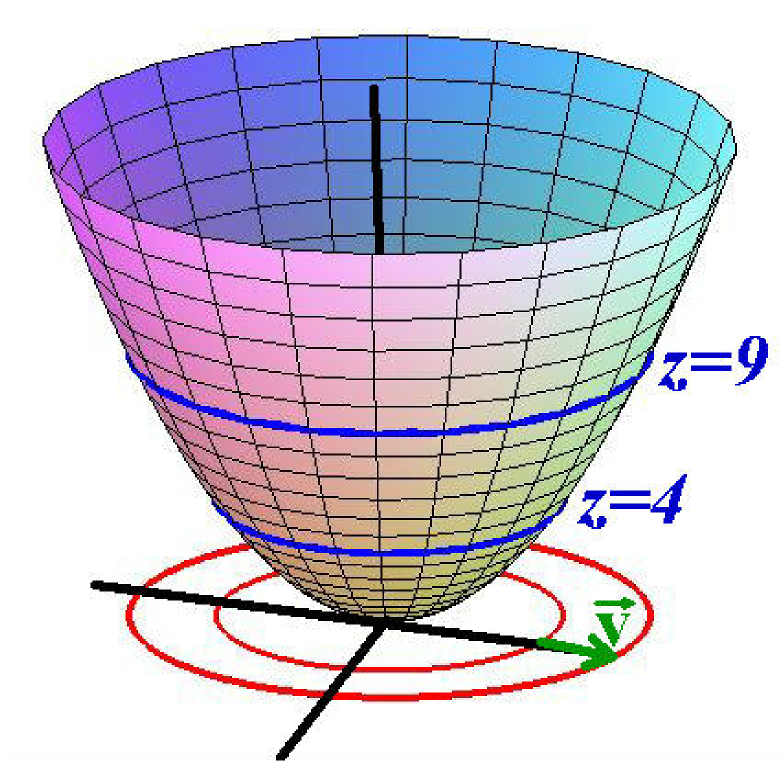


$$z = x^2 + y^2$$

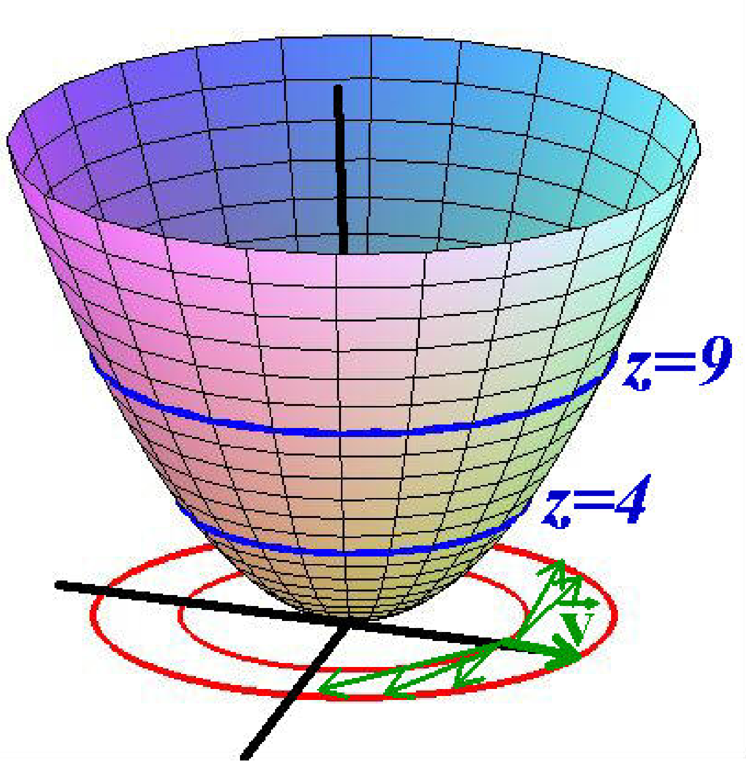
The red curves in the xy plane are both level sets



$$\textbf{Slope} = \frac{\Delta z}{|\vec{\mathbf{v}}|}$$



$$\mathbf{Slope} = \frac{\Delta z}{|\vec{\mathbf{v}}|}$$



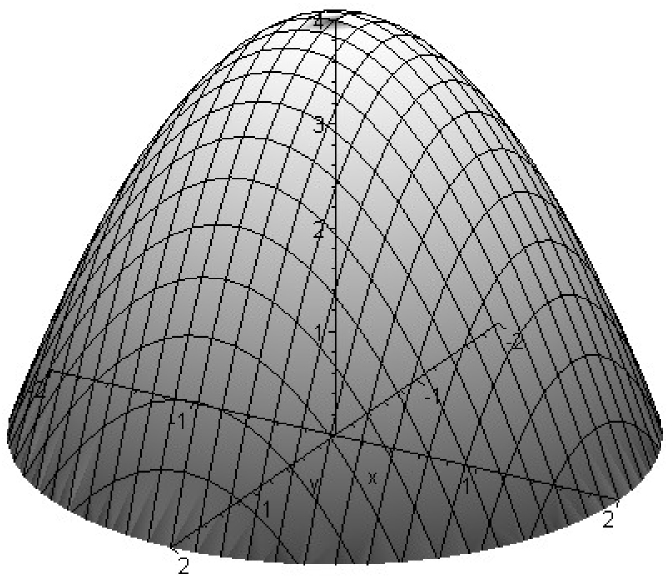
In what direction do we get the biggest value of the directional derivative?

$$\begin{aligned} D_{\vec{\mathbf{v}}} f &= \nabla f \bullet \vec{\mathbf{v}} \\ &= |\nabla f||\vec{\mathbf{v}}| \cos\theta \\ &= |\nabla f| \cos\theta \end{aligned}$$

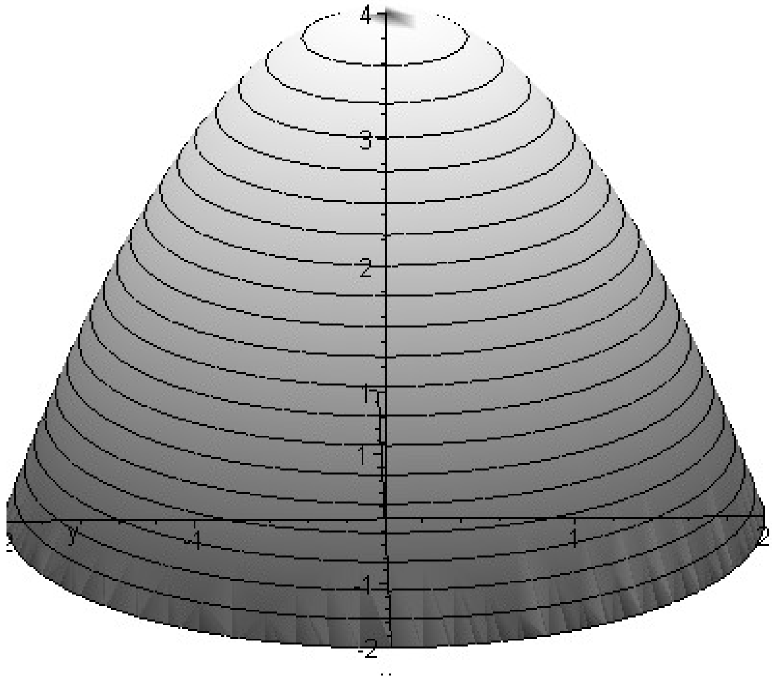
The biggest this gets is when $\theta = 0$ and $\cos\theta = 1$.

$$\text{Maximum value of } D_{\vec{\mathbf{v}}} f = |\nabla f|$$

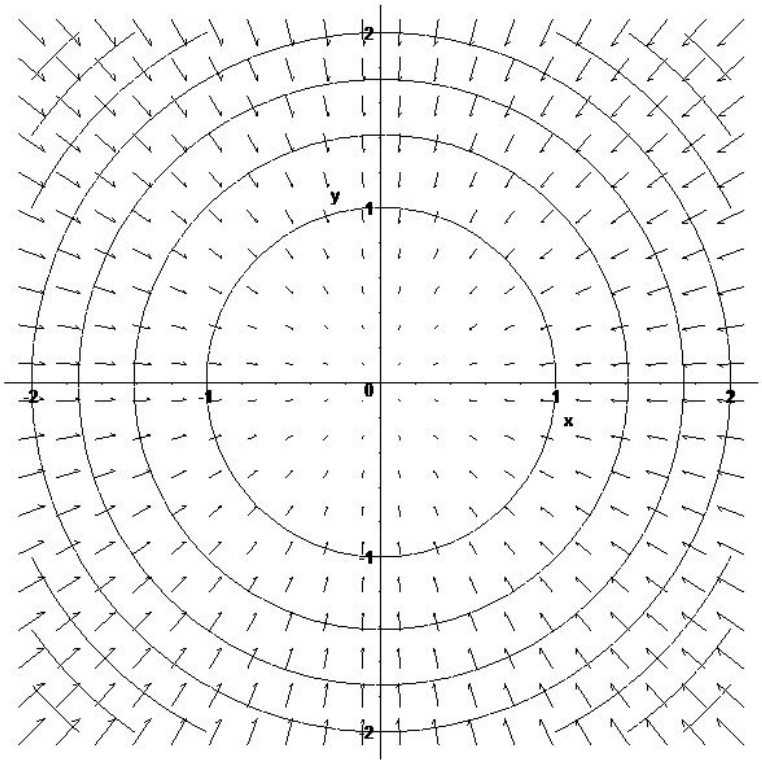
$$f(x,y) = 4 - x^2 - y^2$$

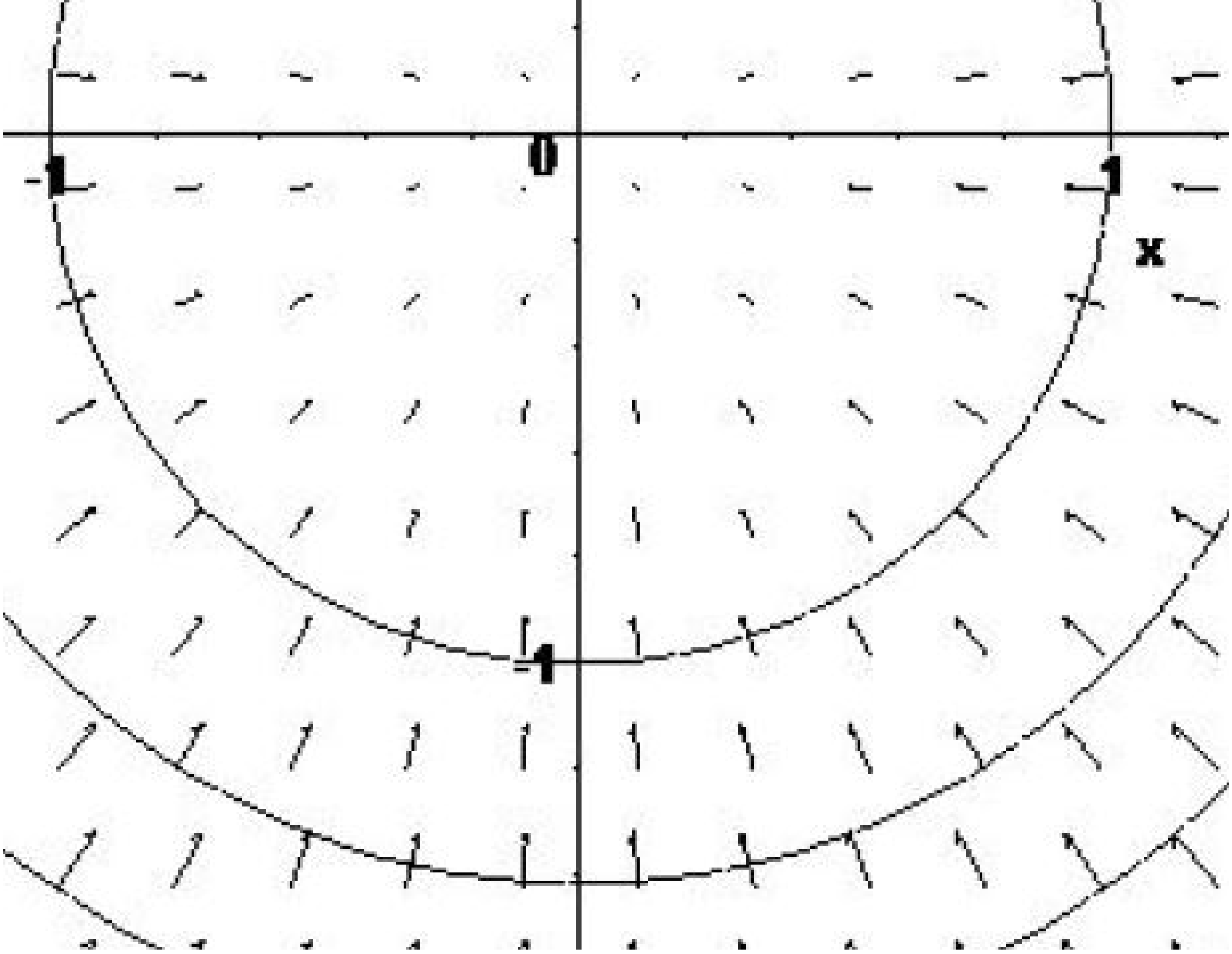


$$f(x, y) = 4 - x^2 - y^2 = C$$



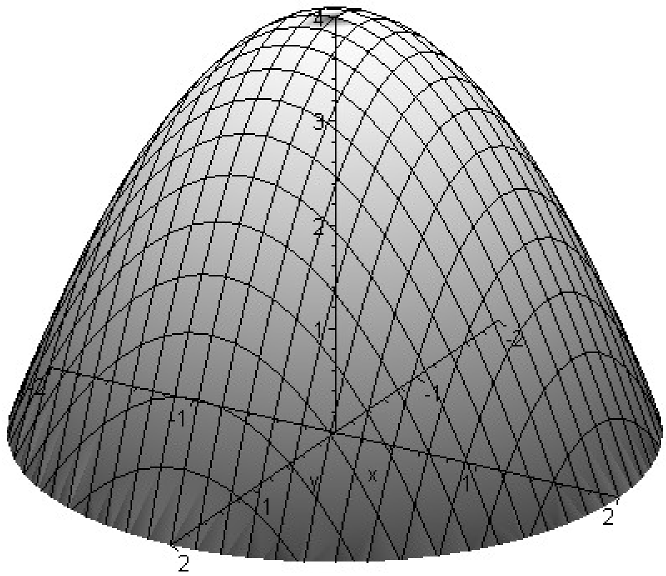
$$f(x,y) = 4 - x^2 - y^2 = C \quad \text{with} \quad \nabla f \quad \text{drawn}$$





0
-1
1
x
-1

$$f(x, y) = 4 - x^2 - y^2$$



$$z = e^y$$

Find the level sets

$$e^y = C$$

$$e^y = C$$

$$\ln\left(e^y\right) = \ln C$$

$$y = \ln C = \text{const}$$

$$e^y = C$$

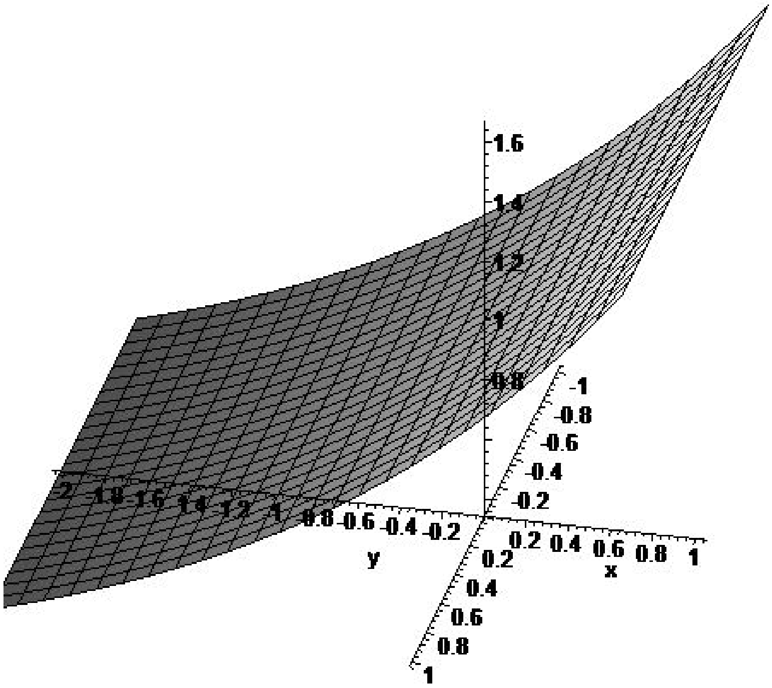
$$\ln\left(e^y\right) = \ln C$$

$$y = \ln C = \text{const}$$

Let's calculate the gradient also:

$$\nabla z = \left\langle \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y} \right\rangle = \langle 0,\ e^y \rangle$$

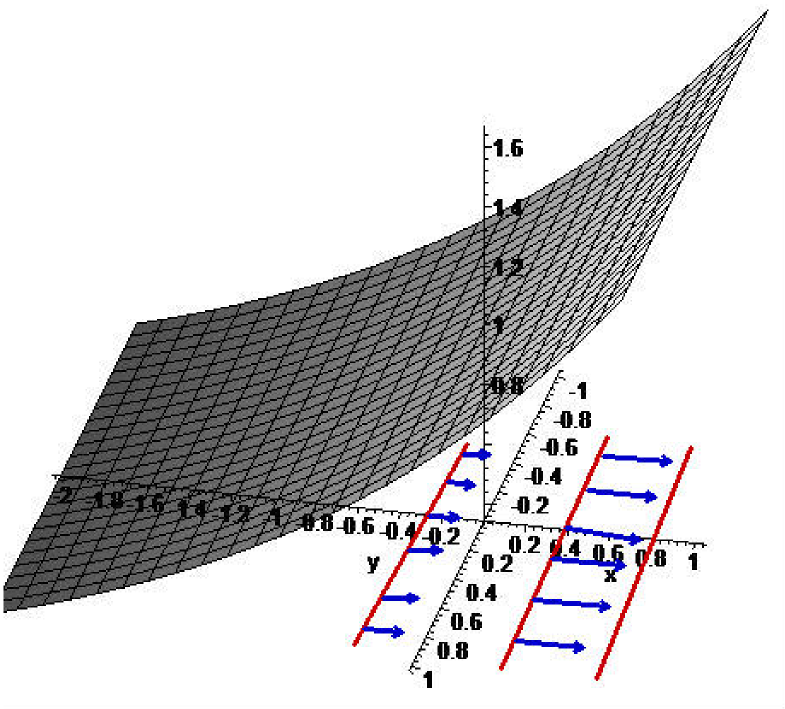
$$z = e^y$$



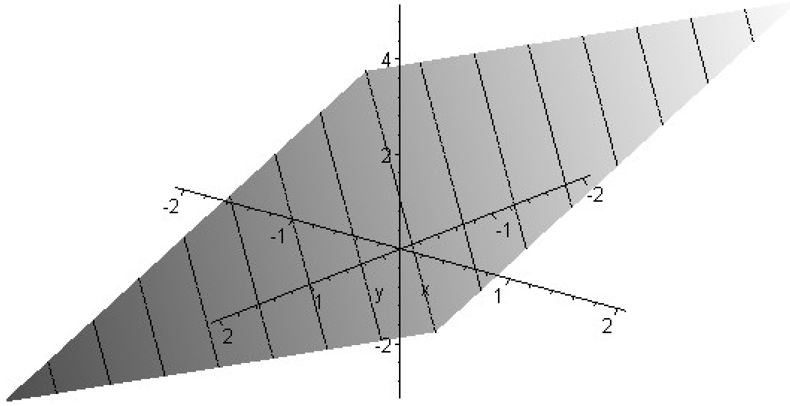
1.6
1.4
1.2
1
0.8
-1
-0.8
-0.6
-0.4
-0.2
-2 -1.8 -1.6 -1.4 -1.2 -1 -0.8 -0.6 -0.4 -0.2
y
0.2 0.4 0.6 0.8 1
x
0.2
0.4
0.6
0.8
1

f

$$\nabla z = e^y \vec{\mathbf{j}}$$

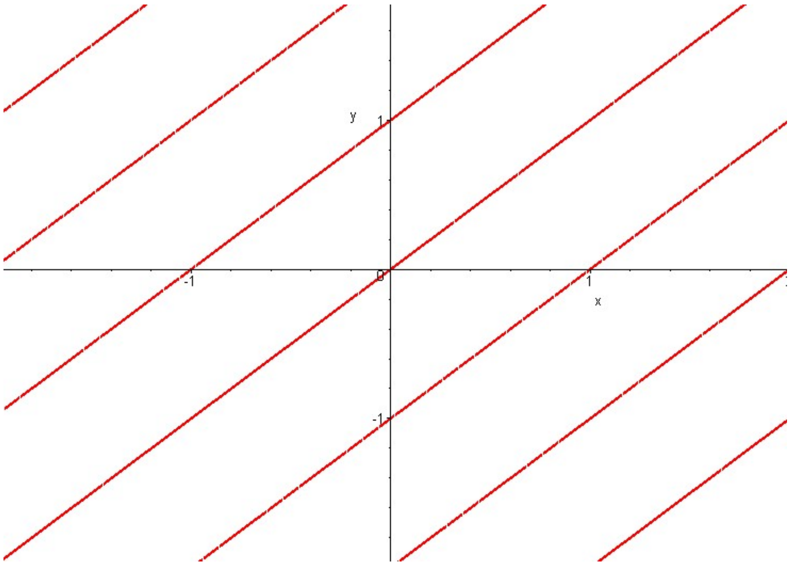


$$f(x, y) = 1 - x + y$$

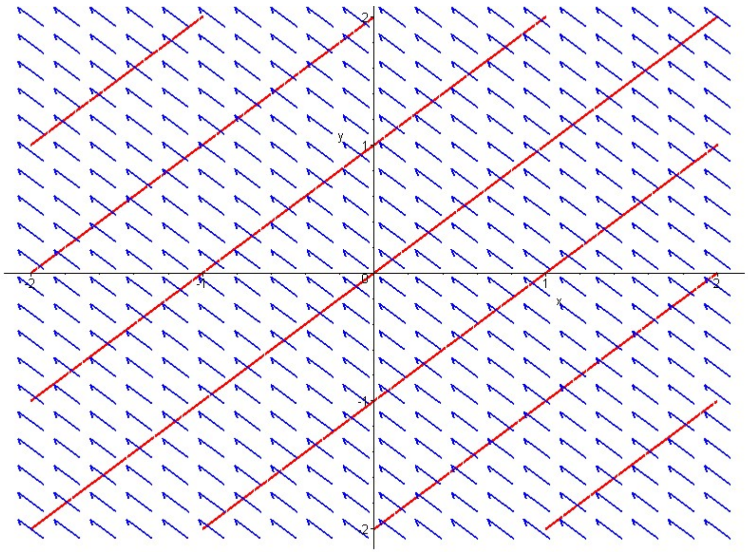


$$f(x, y) = 1 - x + y = C$$

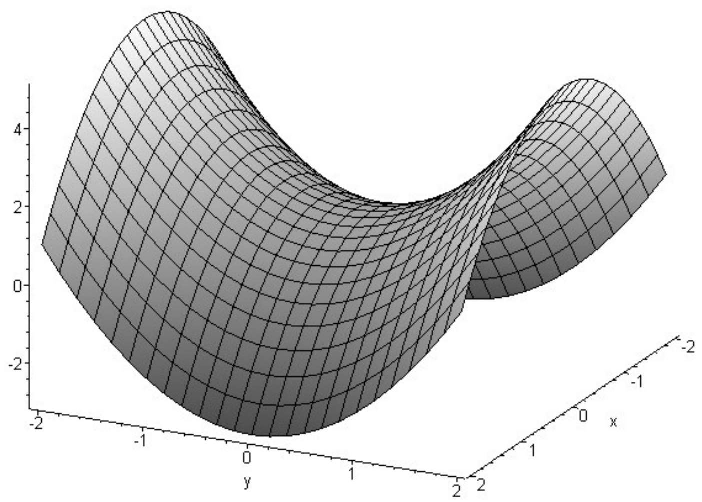
$$y = x + C - 1$$



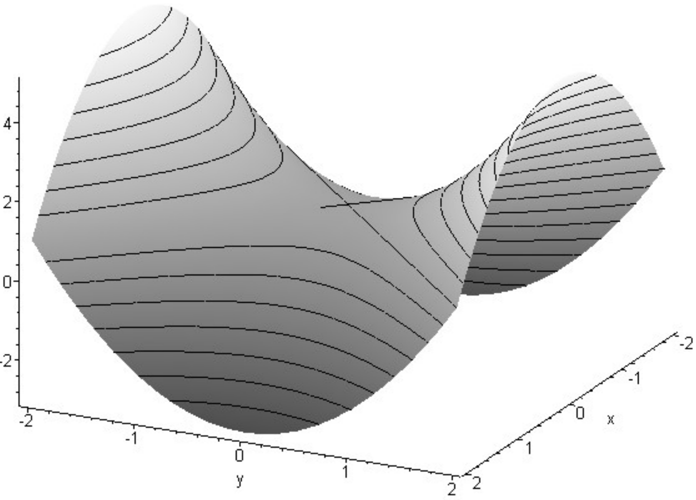
$$f(x,y) = 1 - x + y = C \qquad \nabla f = \langle -1,\ 1 \rangle$$



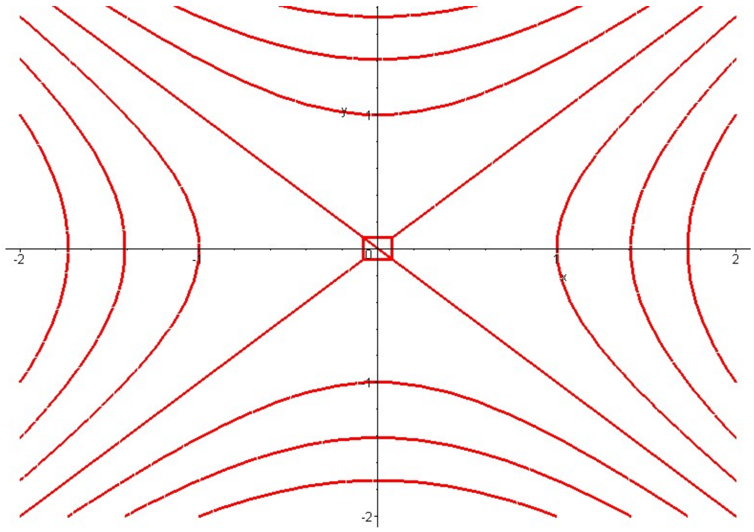
$$f(x, y) = 1 - x^2 + y^2$$



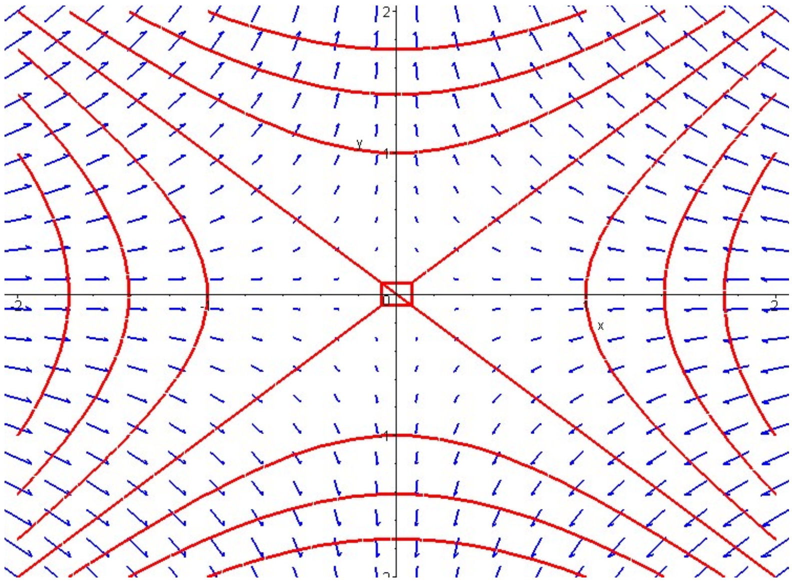
$$f(x, y) = 1 - x^2 + y^2 = C$$



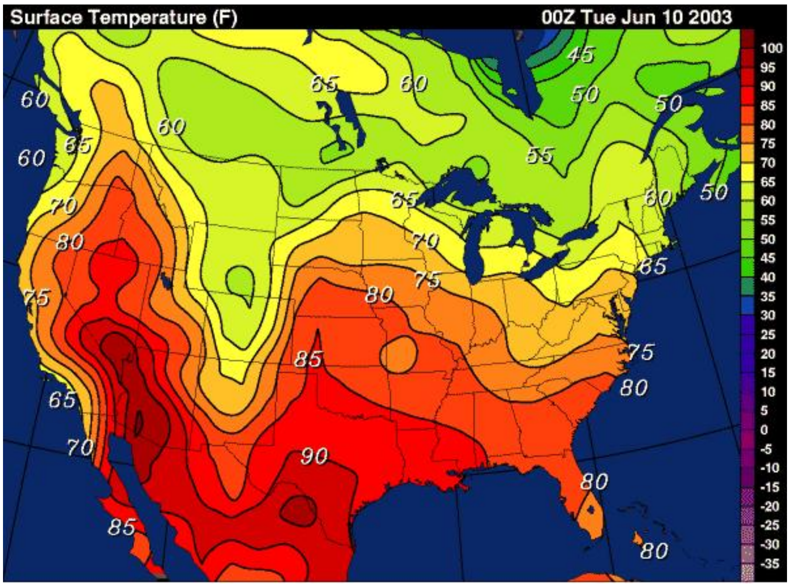
$$f(x,y) = 1 - x^2 + y^2 = C$$



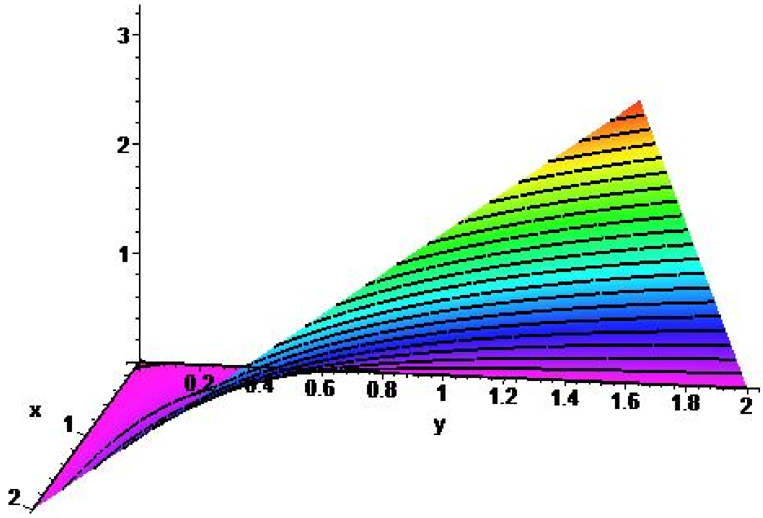
$$f(x,y) = 1 - x^2 + y^2 \qquad \nabla f = \langle -2x,\ 2y \rangle$$



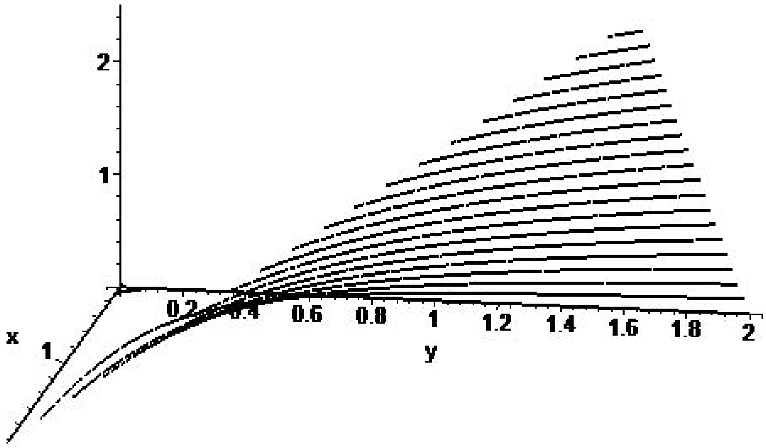
Temperatures



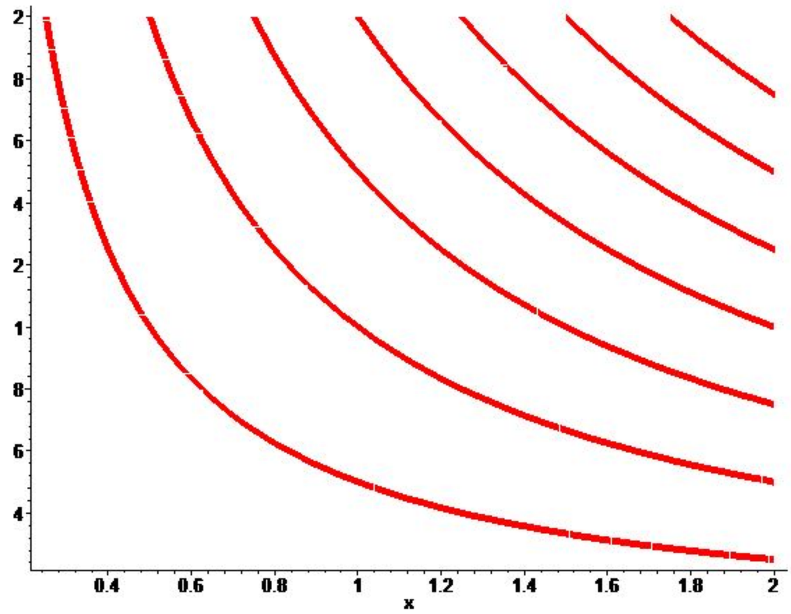
$$f(x, y) = xy$$



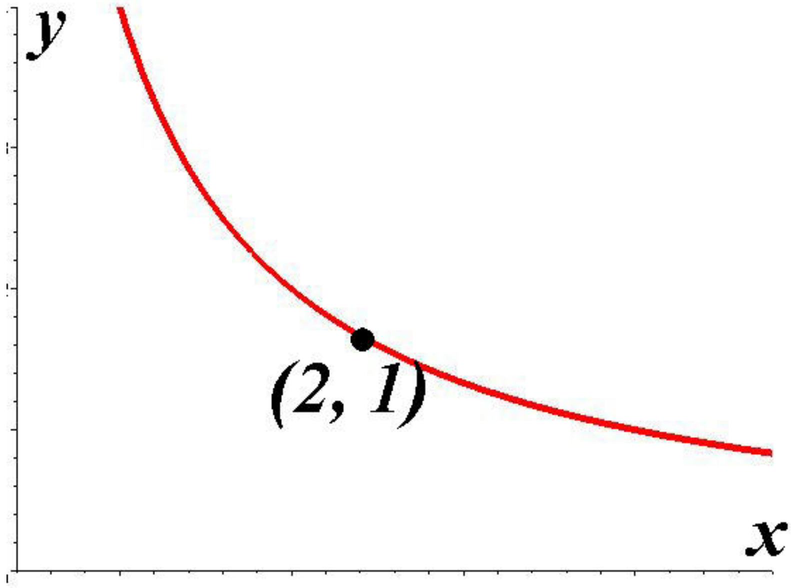
$$f(x, y) = xy$$



$$xy = C$$

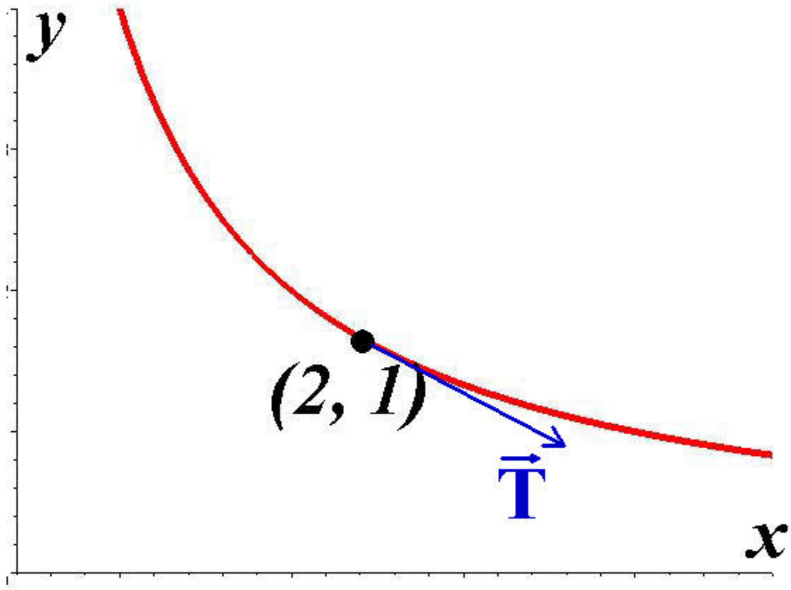


$$xy = 2$$



If $f(x, y) = xy$, find a unit vector $\vec{\mathbf{T}}$ such that:

$$D_{\vec{\mathbf{T}}}\, f(2,\, 1) = 0$$



If $f(x, y) = xy$, find a unit vector $\vec{\mathbf{T}}$ such that:

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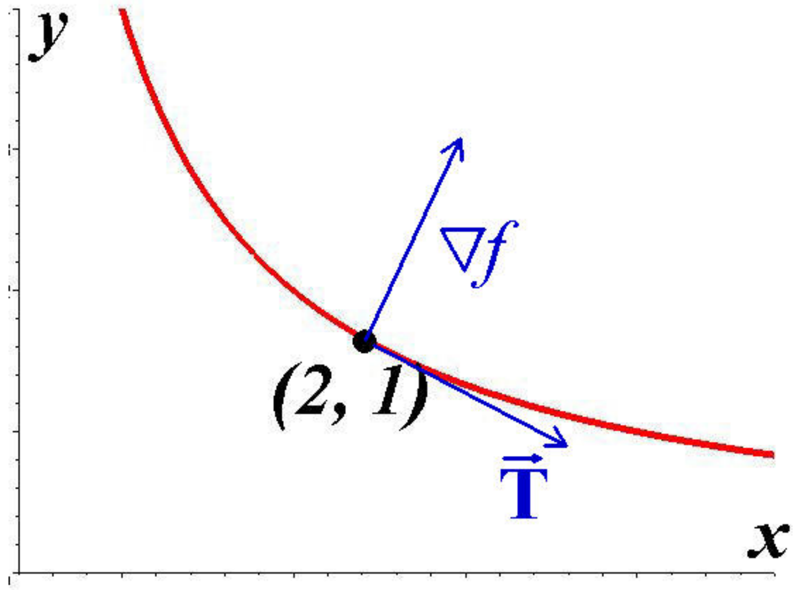

If $f(x, y) = xy$

$$\nabla f = \langle y,\ x \rangle$$

$$\nabla f(2,\ 1) = \langle 1,\ 2 \rangle$$

If $f(x,y) = xy$

$$\nabla f = \langle y,\ x \rangle$$

$$\nabla f(2,\ 1) = \langle 1,\ 2 \rangle$$

We want a vector $\vec{\mathbf{T}} = \langle t_1,\ t_2 \rangle$ that is perpendicular to this.

$$\langle t_1,\ t_2 \rangle \bullet \langle 1,\ 2 \rangle$$

If $f(x, y) = xy$

$$\nabla f = \langle y,\ x \rangle$$

$$\nabla f(2,\ 1) = \langle 1,\ 2 \rangle$$

We want a vector $\vec{\mathbf{T}} = \langle t_1,\ t_2 \rangle$ that is perpendicular to this.

$$\langle t_1,\ t_2 \rangle \bullet \langle 1,\ 2 \rangle$$

$$t_1 + 2t_2 = 0$$

$$t_1 = -2t_2$$

$$t_1 = -2t_2$$

$$\vec{\mathbf{T}} = \langle t_1,\ t_2 \rangle = \langle -2t_2,\ t_2 \rangle$$

We want $\vec{\mathbf{T}}$ to be a unit vector

$$|\langle -2t_2,\ t_2 \rangle| = 1$$

$$t_1 = -2t_2$$

$$\vec{\mathbf{T}} = \langle t_1,\ t_2 \rangle = \langle -2t_2,\ t_2 \rangle$$

We want $\vec{\mathbf{T}}$ to be a unit vector

$$|\langle -2t_2,\ t_2 \rangle| = 1$$

$$\sqrt{4t_2^2 + t_2^2} = 1$$

$$5t_2^2 = 1$$

$$t_2 = \pm\frac{1}{\sqrt{5}}$$

$$t_2 = \pm\frac{1}{\sqrt{5}}$$

$$\vec{\mathbf{T}} = \langle -2t_2,\ t_2 \rangle = \pm\frac{1}{\sqrt{5}}\langle -2,\ 1 \rangle$$