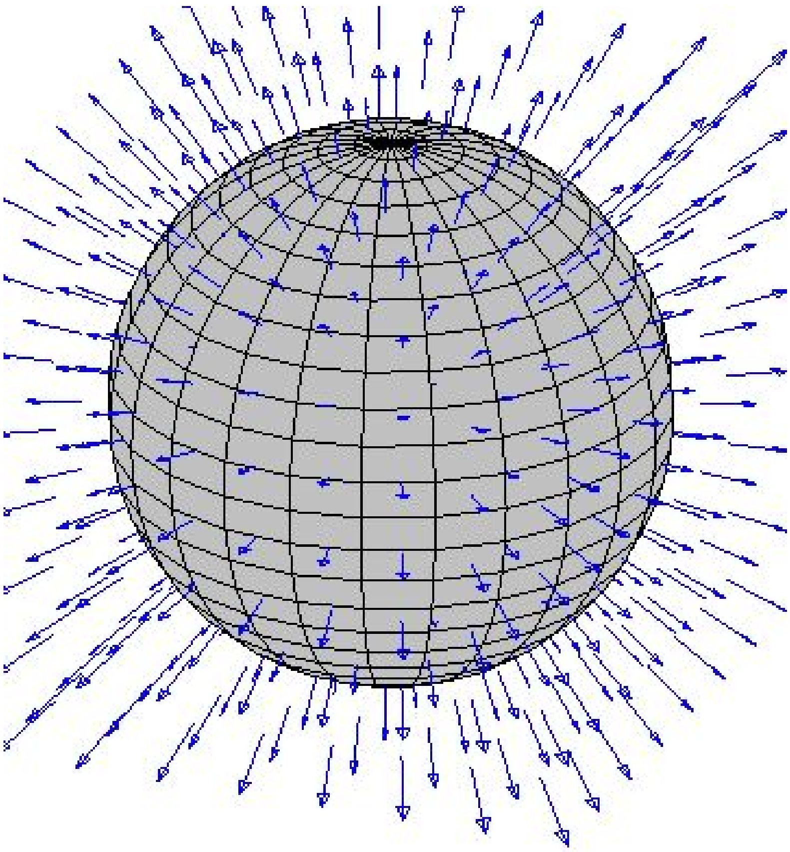
Gradient in Higher Dimension



If $f = f(x, y)$ then:

$$\nabla f = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y} \right\rangle$$

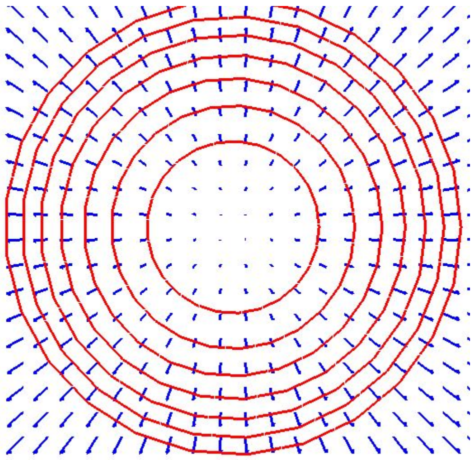
If $f = f(x, y, z)$

$$\nabla f = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right\rangle$$

A level set (or contour) is a set of points $\mathbf{P}$ with the property that $f(\mathbf{P}) = C$

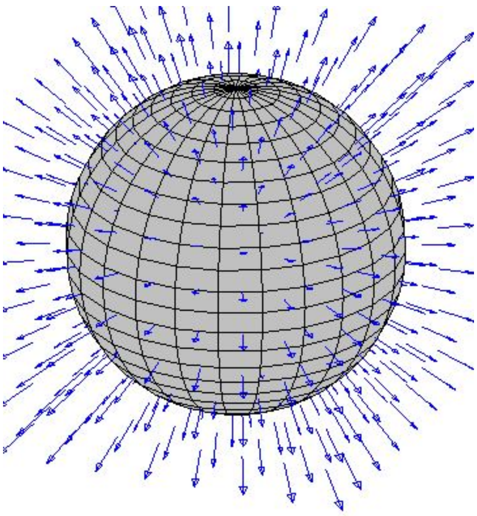
Example: A level set for $f(x, y) = x^2 + y^2$ would be a circle:

$$x^2 + y^2 = C$$



A level set for $f(x, y, z) = x^2 + y^2 + z^2$ would be a sphere:

$$x^2 + y^2 + z^2 = C$$



$$f(x, y, z) = e^{-(x-1)^2 - y^2 - z^2/4}$$

Find the contours

$$f(x, y, z) = C$$

$$f(x,y,z) = e^{-(x-1)^2-y^2-z^2/4}$$

Find the contours

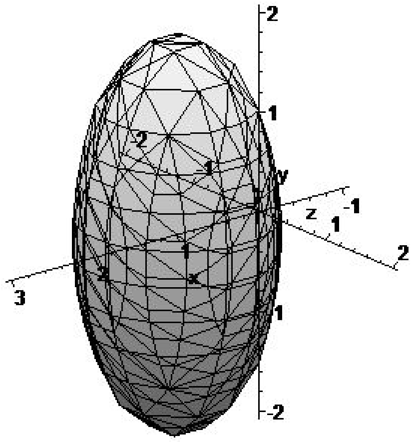
$$e^{-(x-1)^2-y^2-z^2/4} = C$$

$$\ln\left(e^{-(x-1)^2-y^2-z^2/4}\right) = \ln C$$

$$-(x-1)^2 - y^2 - \frac{z^2}{4} = \ln C$$

$$(x-1)^2 + y^2 + \frac{z^2}{4} = -\ln C = \text{constant}$$

$$(x-1)^2 + y^2 + \frac{z^2}{4} = \text{constant}$$



$$f = e^{-(x-1)^2-y^2-z^2/4}$$

$$\frac{\partial f}{\partial x} = -2(x-1)e^{-(x-1)^2-y^2-z^2/4} = -2(x-1)\cdot f$$

$$f = e^{-(x-1)^2 - y^2 - z^2/4}$$

$$\frac{\partial f}{\partial x} = -2(x-1)e^{-(x-1)^2 - y^2 - z^2/4} = -2(x-1) \cdot f$$

Similarly,

$$\frac{\partial f}{\partial y} = -2yf \qquad \frac{\partial f}{\partial z} = -\frac{1}{2}zf$$

$$f = e^{-(x-1)^2-y^2-z^2/4}$$

$$\frac{\partial f}{\partial x} = -2(x-1)e^{-(x-1)^2-y^2-z^2/4} = -2(x-1) \cdot f$$

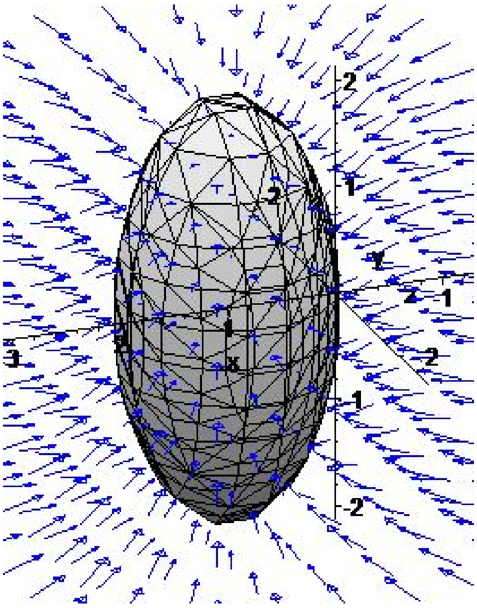
Similarly,

$$\frac{\partial f}{\partial y} = -2yf \qquad \frac{\partial f}{\partial z} = -\frac{1}{2}zf$$

$$\nabla f = \langle -2(x-1)f,\ -2yf,\ -\frac{1}{2}zf \rangle$$

$$f = e^{-(x-1)^2 - y^2 - z^2/4}$$

$$\nabla f = e^{-(x-1)^2 - y^2 - z^2/4} \langle -2(x-1),\ -2y,\ -\frac{1}{2}z \rangle$$



$$f = e^{-(x-1)^2 - y^2 - z^2/4}$$

Let $\vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle 1,\ -1,\ 0\rangle$. Calculate $D_{\vec{\mathbf{v}}} f(2, 0, 0)$

$$f = e^{-(x-1)^2 - y^2 - z^2/4}$$

Let $\vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle 1,\ -1,\ 0\rangle$. Calculate $D_{\vec{\mathbf{v}}} f(2,0,0)$

$$\nabla f = e^{-(x-1)^2 - y^2 - z^2/4}\langle -2(x-1), -2y, -\frac{1}{2}z\rangle$$

$$\nabla f(2,0,0) = e^{-1}\langle -2,0,0\rangle = -\frac{2}{e}\langle 1,0,0\rangle$$

$$f = e^{-(x-1)^2-y^2-z^2/4}$$

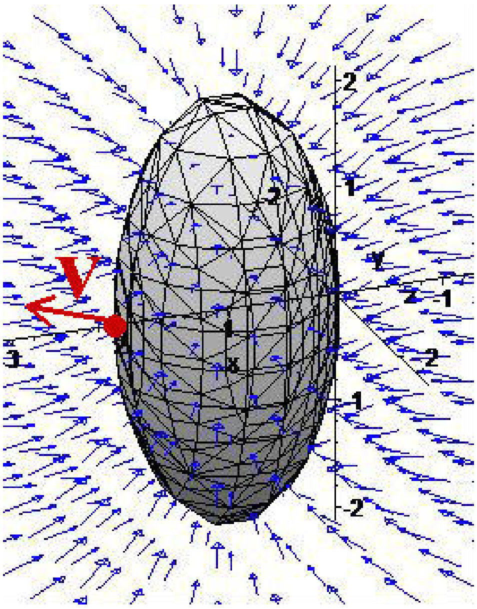
Let $\vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle 1,\ -1,\ 0\rangle$.

$$\nabla f(2,0,0) = -\frac{2}{e}\langle 1,0,0\rangle$$

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(2,0,0) &= \nabla f(2,0,0) \bullet \vec{\mathbf{v}} \\ &= -\frac{2}{e}\langle 1,0,0\rangle \bullet \frac{1}{\sqrt{2}}\langle 1,\ -1,\ 0\rangle \\ &= -\frac{\sqrt{2}}{e} \end{aligned}$$

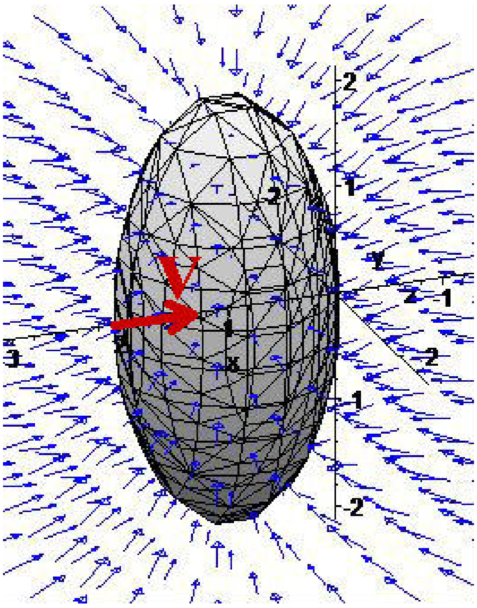
$$f = e^{-(x-1)^2 - y^2 - z^2/4}$$

$\vec{\mathbf{v}} = \frac{1}{\sqrt{2}}\langle 1,\ -1,\ 0\rangle$. $D_{\vec{\mathbf{v}}} f(2,0,0) = -\frac{\sqrt{2}}{e}$



$$f = e^{-(x-1)^2 - y^2 - z^2/4} \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 0 \rangle$$

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(2,0,0) &= \nabla f(2,0,0) \bullet \vec{\mathbf{v}} \\ &= -\frac{2}{e} \langle 1,0,0 \rangle \bullet \langle -1,\ 0,\ 0 \rangle = \frac{2}{e} \end{aligned}$$



Any surface given by an equation of the form:

$$z = f(x, y)$$

can always be written as an equation of the form:

$$G(x, y, z) = \text{constant}$$

Example:

$$z = 1 - (x + 1)^2 - (y + 1)^2$$

implies that

$$(x + 1)^2 + (y + 1)^2 + z = 1$$

Any surface described by the equation:

$$G(x, y, z) = C$$

will be a level set of the function

$$w = G(x, y, z)$$

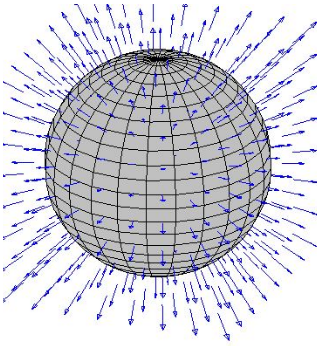
Any surface described by the equation:

$$G(x, y, z) = C$$

will be a level set of the function

$$w = G(x, y, z)$$

∇G will be perpendicular to this surface.



Example

Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x+1)^2 - (y+1)^2$ at the point $(-1,\ 0,\ 0)$.

$$z = z_0 + \frac{\partial z}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial z}{\partial y}(x_0, y_0) \cdot (y - y_0)$$

Example

Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x + 1)^2 - (y + 1)^2$ at the point $(-1,\ 0,\ 0)$.

$$\vec{\mathbf{n}} \bullet (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$$

$$z = 1 - (x+1)^2 - (y+1)^2$$

$$(x+1)^2 + (y+1)^2 + z = 1$$

Let $G(x, y, z) = (x+1)^2 + (y+1)^2 + z$. The paraboloid is a level set of $G(x, y, z)$

$$\nabla G(x, y, z) = \langle 2(x+1),\ 2(y+1),\ 1\rangle$$

$$\nabla G(-1,\ 0,\ 0) = \langle 0,\ 2,\ 1\rangle$$

This will be perpendicular to the paraboloid.

Example

Find the equation of the plane that is tangent to the paraboloid $z = 1 - (x+1)^2 - (y+1)^2$ at the point $(-1,\ 0,\ 0)$.

$$\vec{\mathbf{n}} \bullet (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$$

$$\langle 0,\ 2,\ 1\rangle \bullet \langle x-(-1),\ y-0,\ z-0\rangle = 0$$

$$2y + z = 0$$