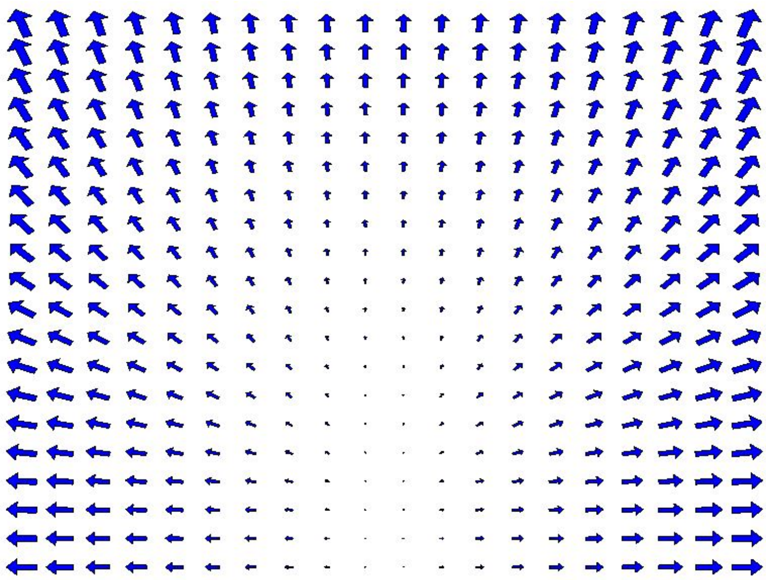
Calculation of Potential Functions

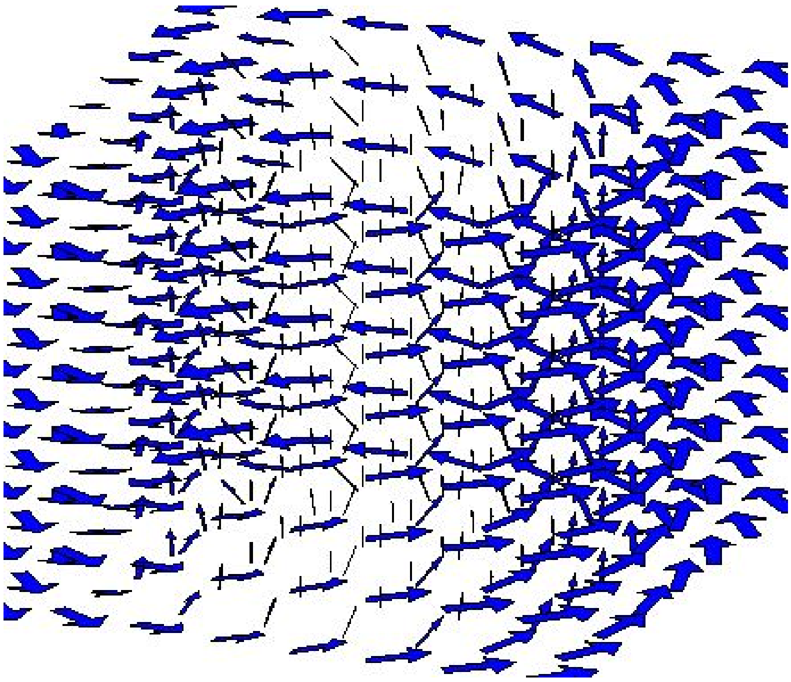
$$\vec{\mathbf{F}} = \nabla \phi$$

$$\phi(x, y) = x^2 + y^3$$

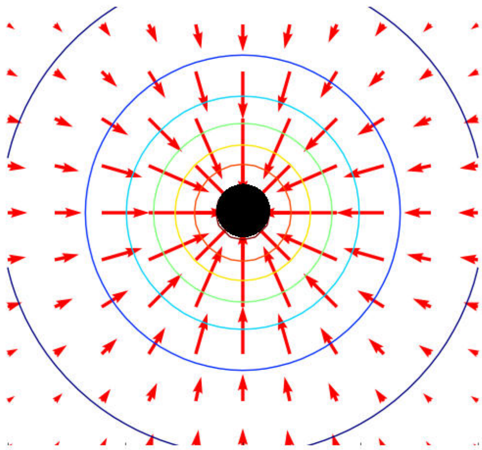
$$\nabla\phi = \langle 2x,\ 3y^2 \rangle$$



$$\vec{\mathbf{F}} = \langle -y,\ x, 1 \rangle$$



$$\vec{\mathbf{F}} = -\frac{G m_1 m_2}{|\vec{\mathbf{r}}|^3}\,\vec{\mathbf{r}} = \nabla \phi$$

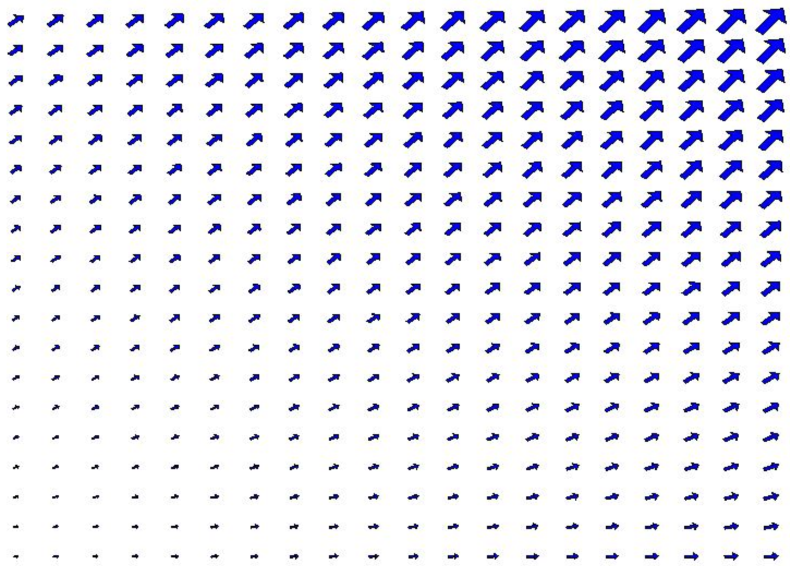


If $\vec{\mathbf{F}}$ is the gradient of a scalar-valued function ϕ then ϕ is called a *potential function*

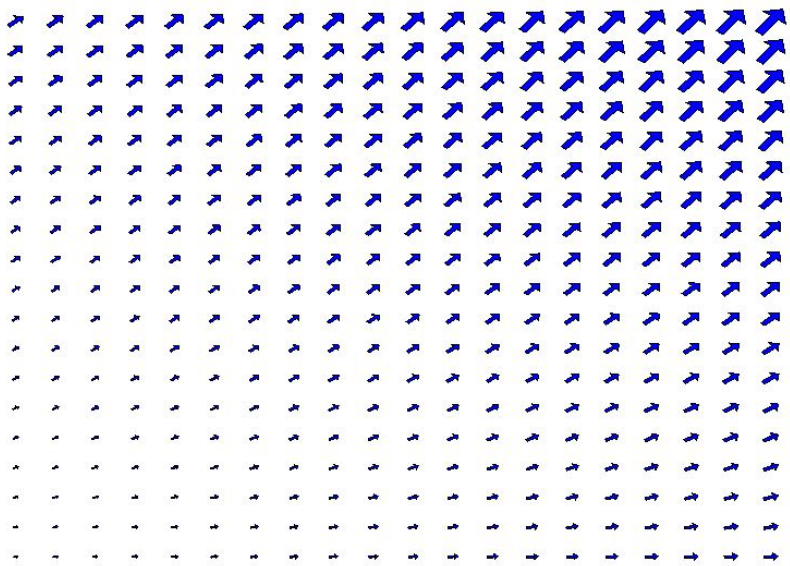
$$\vec{\mathbf{F}} = \nabla \phi$$

$$\vec{\mathbf{F}} = \langle 3y^2 - 6x,\ 6xy \rangle$$

Find a function ϕ so that $\vec{\mathbf{F}} = \nabla\phi$



$$\vec{\mathbf{F}} = \langle 3y^2 - 6x,\ 6xy \rangle = \left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y} \right\rangle$$



$$\vec{\mathbf{F}} = \langle 3y^2 - 6x,\ 6xy \rangle = \left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y} \right\rangle$$

$$\frac{\partial \phi}{\partial x} = 3y^2 - 6x \qquad \frac{\partial \phi}{\partial y} = 6xy$$

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$$\text{If } \frac{\partial \phi}{\partial y} = 6xy \text{ then } \phi = \int 6xy \, dy$$

$$\frac{\partial \phi}{\partial y} = 6xy$$

$$\phi = \int 6xy\,dy \quad \text{(where } x \text{ is held constant)}$$

$$\frac{\partial \phi}{\partial y} = 6xy$$

$$\begin{aligned}\phi &= \int 6xy\, dy \quad \text{(where } x \text{ is held constant)} \\ &= 3xy^2 + C\end{aligned}$$

$$\frac{\partial}{\partial y}\left(3xy^2 + C\right) = 6xy$$

$$\frac{\partial}{\partial y}\left(3xy^2 + C\right) = 6xy$$

$$\frac{\partial}{\partial y}\left(3xy^2 + \sin x\right) = 6xy$$

$$\frac{\partial}{\partial y}\left(3xy^2 + e^x\right) = 6xy$$

$$\frac{\partial}{\partial y}\left(3xy^2 + \sqrt{x}\right) = 6xy$$

$$\frac{\partial \phi}{\partial y} = 6xy$$

$$\begin{aligned}\phi &= \int 6xy\, dy \quad \text{(where } x \text{ is held constant)} \\ &= 3xy^2 + f(x)\end{aligned}$$

$$\phi = 3xy^2 + f(x)$$

We also have the condition that $\frac{\partial \phi}{\partial x} = 3y^2 - 6x$

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$$3y^2 + f'(x) = 3y^2 - 6x$$

$$f'(x) = -6x$$

$$f(x) = -3x^2 + C$$

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$$\phi = 3xy^2 - 3x^2 + C$$

$\phi = 3xy^2 - 3x^2 + C$ has the property that $\nabla\phi = \langle 3y^2 - 6x,\ 6xy \rangle$ for any value of C, so if $C = 0$,

$$\phi = 3xy^2 - 3x^2$$

$$\vec{\mathbf{F}} = \langle 2x \sin y,\ x^2 \cos y \rangle$$

Find $\phi(x, y)$ so that $\vec{\mathbf{F}} = \nabla \phi$

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If $\frac{\partial \phi}{\partial x} = 2x \sin y$ then:

$$\phi = \int 2x \sin y \, dx$$

$$\frac{\partial \phi}{\partial x} = 2x \sin y \qquad\qquad \frac{\partial \phi}{\partial y} = x^2 \cos y$$

If $\frac{\partial \phi}{\partial x} = 2x \sin y$ then:

$$\phi = \int 2x \sin y \, dx = x^2 \sin y + f(y)$$

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$$x^2 \cos y + f'(y) = x^2 \cos y$$

This implies $f'(y) = 0$, so $f(y)$ can be any constant

$$\phi(x, y) = x^2 \sin y + C$$

If C can be any constant, let's take $C = 0$

$$\phi(x, y) = x^2 \sin y$$

$$\vec{\mathbf{F}} = \langle e^y + 2e^x,\ xe^y + ze^y,\ e^y \rangle$$

Find $\phi(x, y, z)$ so that $\vec{\mathbf{F}} = \nabla\phi$

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$$\phi = \int e^y \, dz$$

$$\vec{\mathbf{F}} = \langle e^y + 2e^x,\ xe^y + ze^y,\ e^y \rangle$$

Find $\phi(x, y, z)$ so that $\vec{\mathbf{F}} = \nabla\phi$

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$$\phi = \int e^y\, dz = e^y z + f(x, y)$$

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$$\frac{\partial}{\partial y} \left(e^y z + f(x, y) \right) = xe^y + ze^y$$

$$e^y z + \frac{\partial f}{\partial y}(x, y) = xe^y + ze^y$$

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$$e^y z + \frac{\partial f}{\partial y}(x, y) = xe^y + ze^y$$

$$\frac{\partial f}{\partial y}(x, y) = xe^y$$

$$\frac{\partial f}{\partial y}(x, y) = xe^y$$

$$f(x, y) = \int xe^y \, dy = xe^y + g(x)$$

$$\phi(x, y, z) = e^y z + f(x, y) = e^y z + xe^y + g(x)$$

We still have one more condition:

$$\frac{\partial \phi}{\partial x} = e^y + 2e^x$$

$$\phi(x, y, z) = e^y z + xe^y + g(x)$$

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$$\frac{\partial \phi}{\partial x} = e^y + 2e^x$$

$$\frac{\partial}{\partial x}\left(e^y z + xe^y + g(x)\right) = e^y + 2e^x$$

$$e^y + g'(x) = e^y + 2e^x$$

$$g'(x) = 2e^x$$

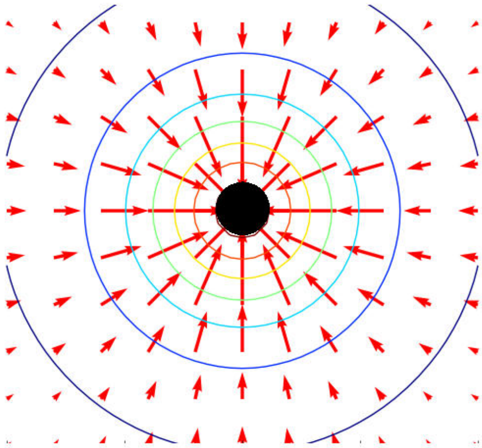
$$g(x) = 2e^x$$

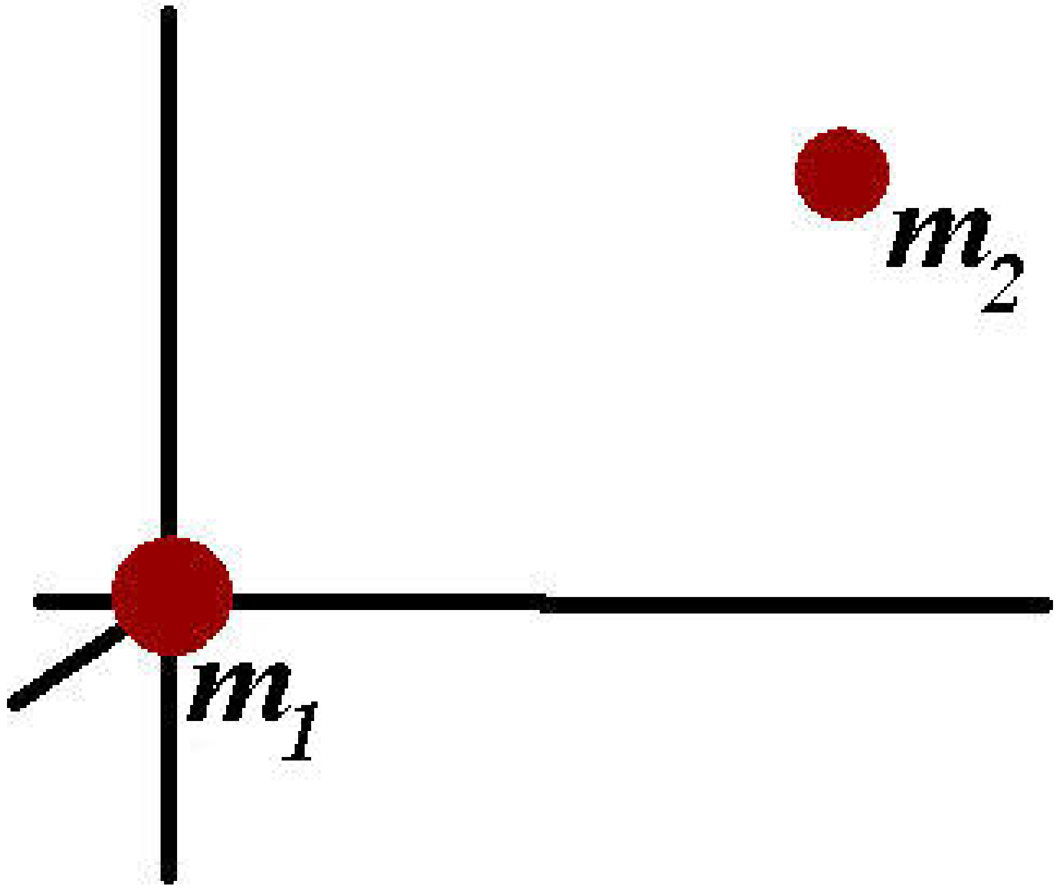
$$\phi(x, y, z) = e^y z + xe^y + g(x) = e^y z + xe^y + 2e^x$$

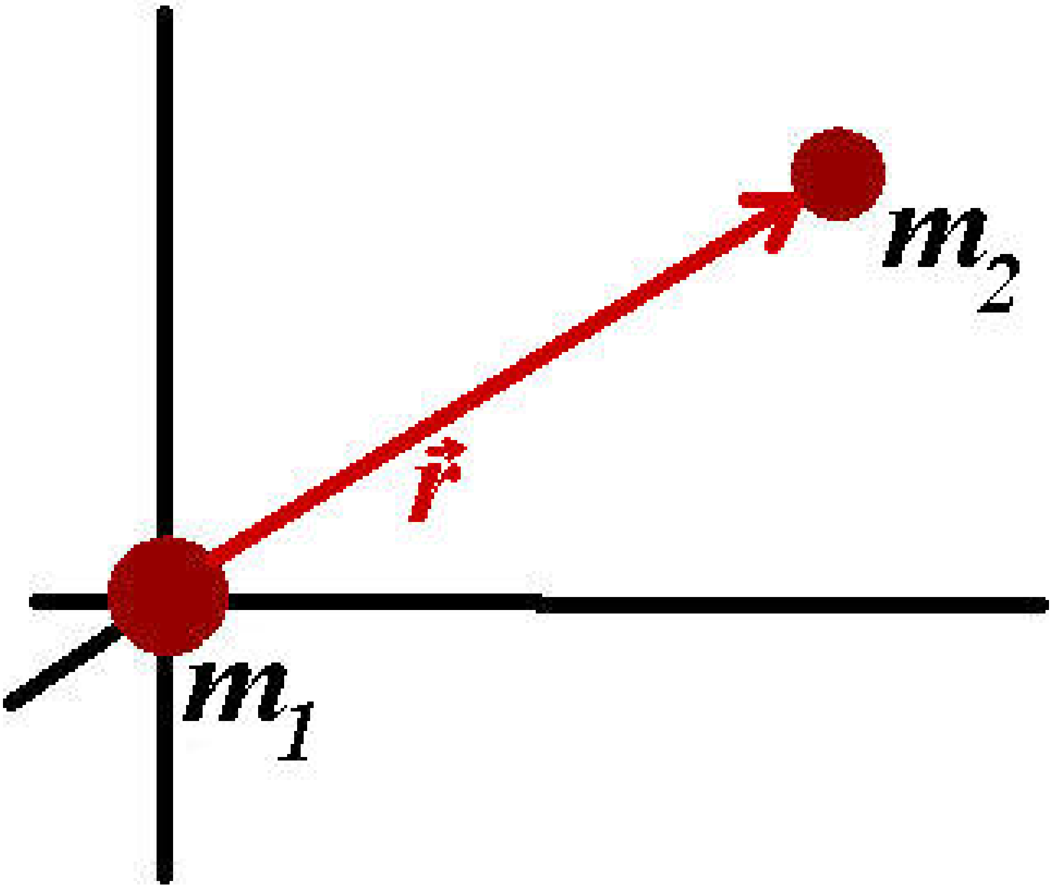
Check:

$$\nabla\phi = \langle e^y + 2e^x,\ e^y z + xe^y,\ e^y \rangle = \vec{\mathbf{F}}$$

Let $\vec{\mathbf{F}}$ be the force of gravity.
Find the potential function ϕ.

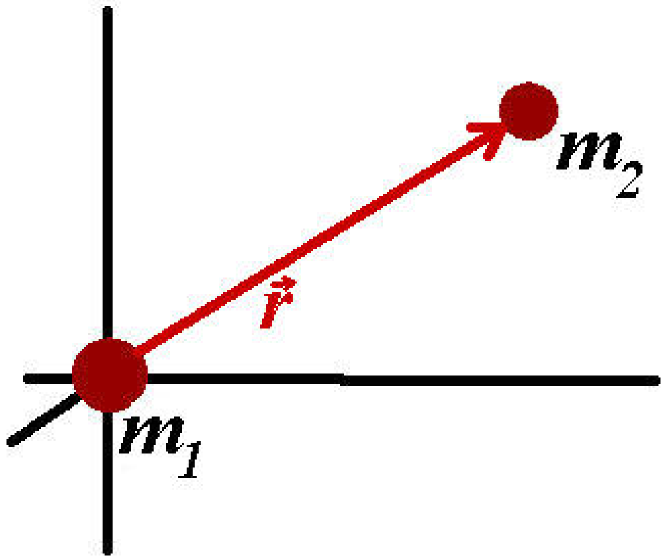



m_2
m_1

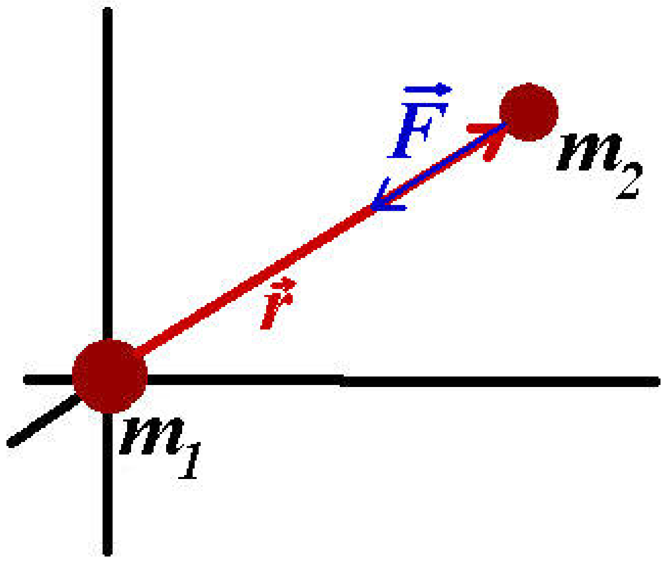


m2
r
m1

$$|\vec{\mathbf{F}}| = \frac{Gm_1m_2}{|\vec{\mathbf{r}}|^2}$$



$$|\vec{\mathbf{F}}| = \frac{Gm_1m_2}{|\vec{\mathbf{r}}|^2} \qquad \frac{\vec{\mathbf{F}}}{|\vec{\mathbf{F}}|} = -\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$$



$$|\vec{\mathbf{F}}| = \frac{Gm_1m_2}{|\vec{\mathbf{r}}|^2} \qquad \frac{\vec{\mathbf{F}}}{|\vec{\mathbf{F}}|} = -\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$$

$$\vec{\mathbf{F}} = |\vec{\mathbf{F}}| \left(-\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} \right)$$

$$|\vec{\mathbf{F}}| = \frac{Gm_1m_2}{|\vec{\mathbf{r}}|^2} \qquad \frac{\vec{\mathbf{F}}}{|\vec{\mathbf{F}}|} = -\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$$

$$\begin{aligned}
\vec{\mathbf{F}} &= |\vec{\mathbf{F}}| \left(-\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}\right) \\
&= \left(\frac{Gm_1m_2}{|\vec{\mathbf{r}}|^2}\right) \left(-\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}\right) \\
&= -\frac{Gm_1m_2}{|\vec{\mathbf{r}}|^3}\vec{\mathbf{r}}
\end{aligned}$$

$$\vec{\mathbf{F}} = -\frac{Gm_1m_2}{|\vec{\mathbf{r}}|^3}\vec{\mathbf{r}}$$

If $\vec{\mathbf{r}} = \langle x,\ y\rangle$ then:

$$\vec{\mathbf{F}} = -\frac{Gm_1m_2}{\left(x^2+y^2\right)^{3/2}}\langle x,\ y\rangle$$

$$\vec{\mathbf{F}} = -\frac{Gm_1m_2}{|\vec{\mathbf{r}}|^3}\vec{\mathbf{r}}$$

If $\vec{\mathbf{r}} = \langle x,\ y\rangle$ then:

$$\vec{\mathbf{F}} = \left\langle -\frac{Gm_1m_2x}{\left(x^2+y^2\right)^{3/2}},\ -\frac{Gm_1m_2y}{\left(x^2+y^2\right)^{3/2}} \right\rangle$$

$$\vec{\mathbf{F}} = \left\langle -\frac{Gm_1m_2x}{\left(x^2+y^2\right)^{3/2}},\ -\frac{Gm_1m_2y}{\left(x^2+y^2\right)^{3/2}} \right\rangle$$

Find ϕ so that $\vec{\mathbf{F}} = \nabla\phi$

$$\frac{\partial\phi}{\partial x} = -\frac{Gm_1m_2x}{\left(x^2+y^2\right)^{3/2}} \qquad \frac{\partial\phi}{\partial y} = -\frac{Gm_1m_2y}{\left(x^2+y^2\right)^{3/2}}$$

$$\begin{aligned}
\phi &= \int -\frac{Gm_1m_2x}{\left(x^2+y^2\right)^{3/2}}\,dx \\
&= -Gm_1m_2\int \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx
\end{aligned}$$

$$
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\phi &= \int -\frac{Gm_1m_2x}{\left(x^2+y^2\right)^{3/2}}\,dx \\
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&= -Gm_1m_2\int \frac{1}{2}u^{-3/2}\,du \qquad \text{where } u = x^2+y^2
\end{aligned}
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&= Gm_1m_2u^{-1/2} + f(y)
\end{aligned}
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&= Gm_1m_2u^{-1/2} + f(y) \\
&= Gm_1m_2\left(x^2+y^2\right)^{-1/2} + f(y)
\end{aligned}
$$

$$\phi = \frac{Gm_1m_2}{(x^2+y^2)^{1/2}} + f(y)$$

We still have the condition that $\frac{\partial \phi}{\partial y} = -\frac{Gm_1m_2y}{(x^2+y^2)^{3/2}}$

$$\frac{\partial}{\partial y}\left(\frac{Gm_1m_2}{(x^2+y^2)^{1/2}} + f(y)\right) = -\frac{Gm_1m_2y}{(x^2+y^2)^{3/2}}$$

$$\phi = \frac{Gm_1m_2}{\left(x^2+y^2\right)^{1/2}} + f(y)$$

We still have the condition that $\frac{\partial \phi}{\partial y} = -\frac{Gm_1m_2y}{\left(x^2+y^2\right)^{3/2}}$

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$$-\frac{Gm_1m_2y}{\left(x^2+y^2\right)^{3/2}} + f'(y) = -\frac{Gm_1m_2y}{\left(x^2+y^2\right)^{3/2}}$$

$$\phi = \frac{Gm_1m_2}{\left(x^2+y^2\right)^{1/2}} + f(y)$$

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$$-\frac{Gm_1m_2}{\left(x^2+y^2\right)^{3/2}} + f'(y) = -\frac{Gm_1m_2y}{\left(x^2+y^2\right)^{3/2}}$$

So $f'(y) = 0$ and $f(y)$ can be any constant

$$\phi = \frac{G m_1 m_2}{\left(x^2 + y^2\right)^{1/2}} = \frac{G m_1 m_2}{\sqrt{x^2 + y^2}}$$

If $\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle$ then:

$$\phi(x, y, z) = \frac{Gm_1m_2}{\sqrt{x^2 + y^2 + z^2}} = \frac{Gm_1m_2}{|\vec{\mathbf{r}}|}$$