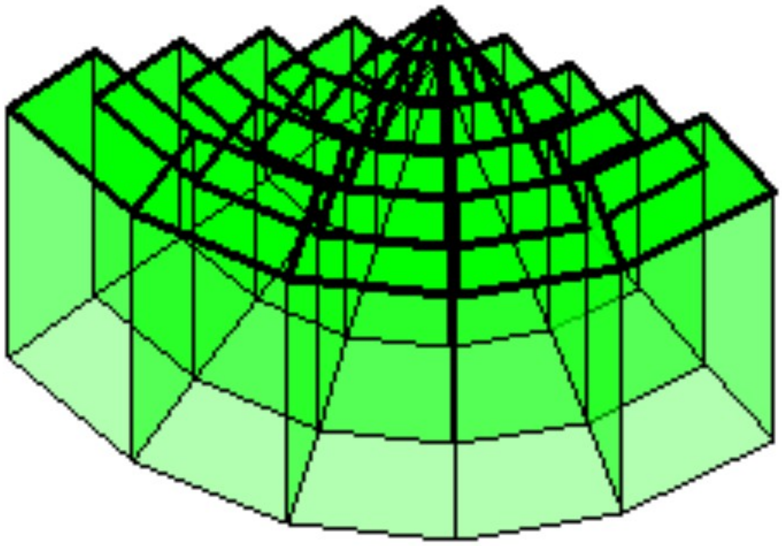
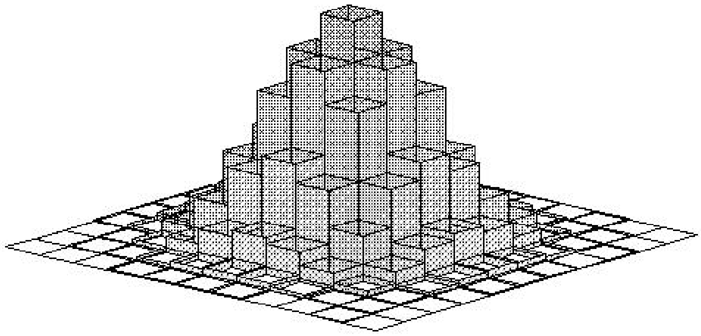
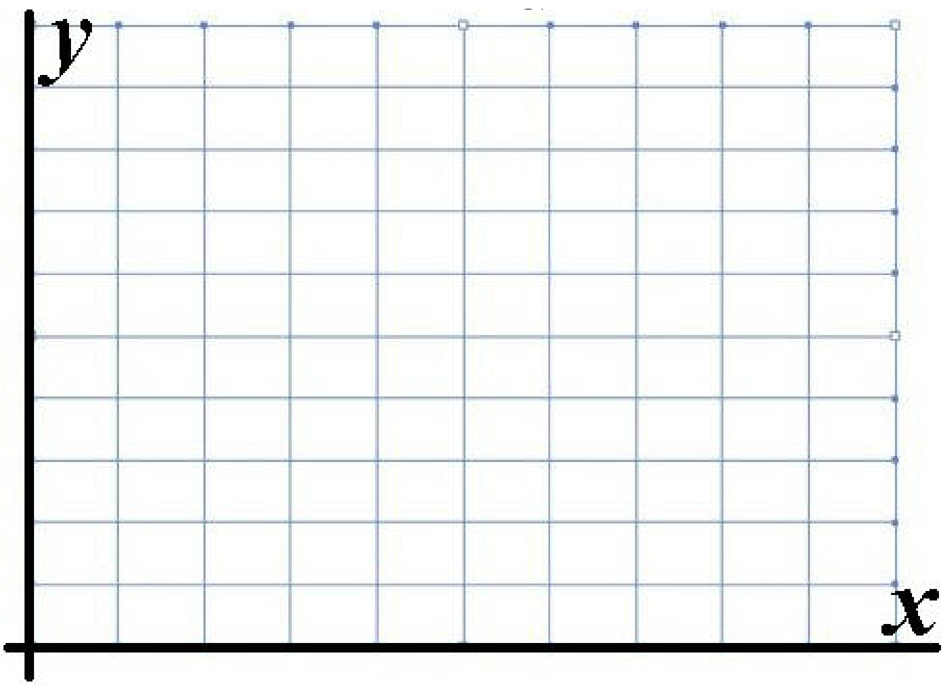
Double Integrals in Polar Coordinates

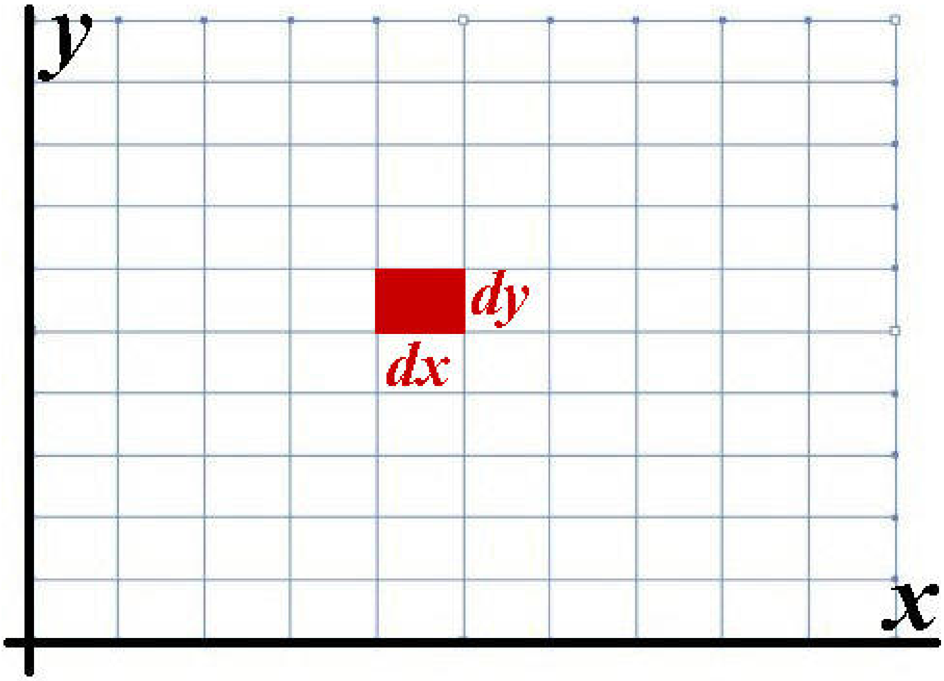


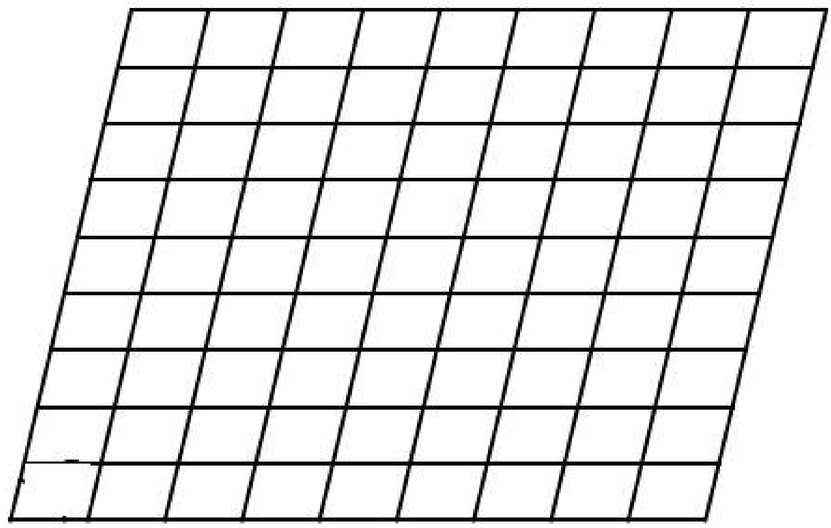
$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

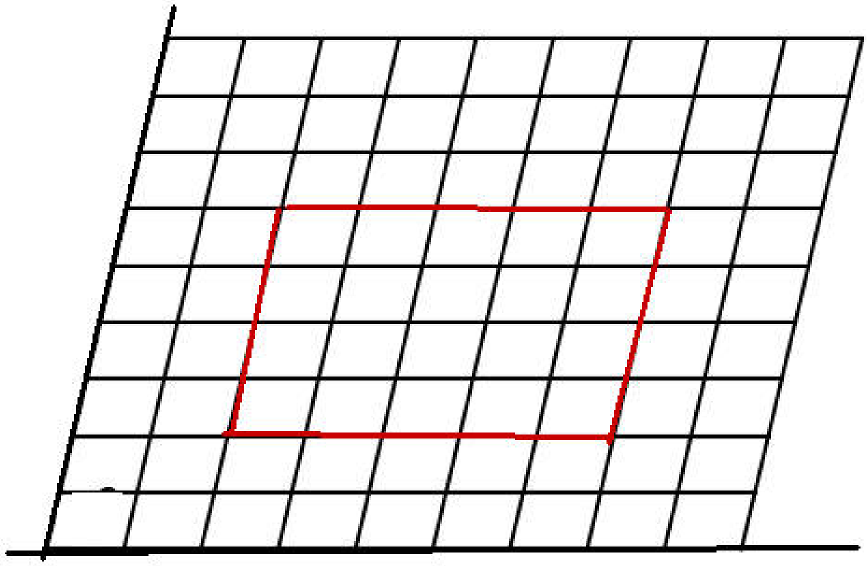


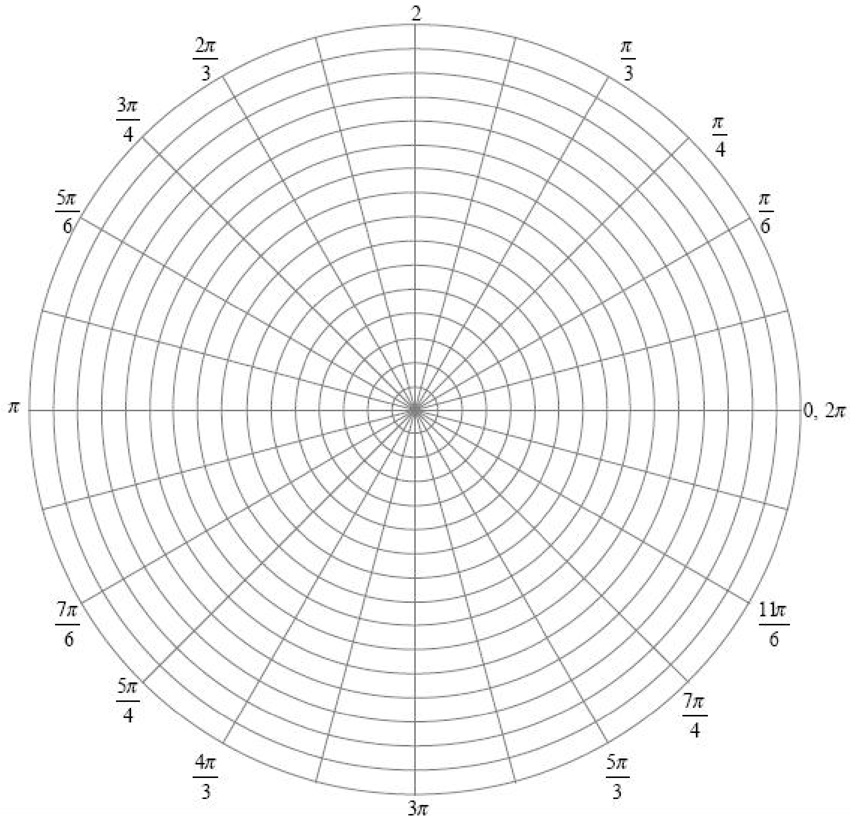

y
x

$$dA = dx\,dy$$

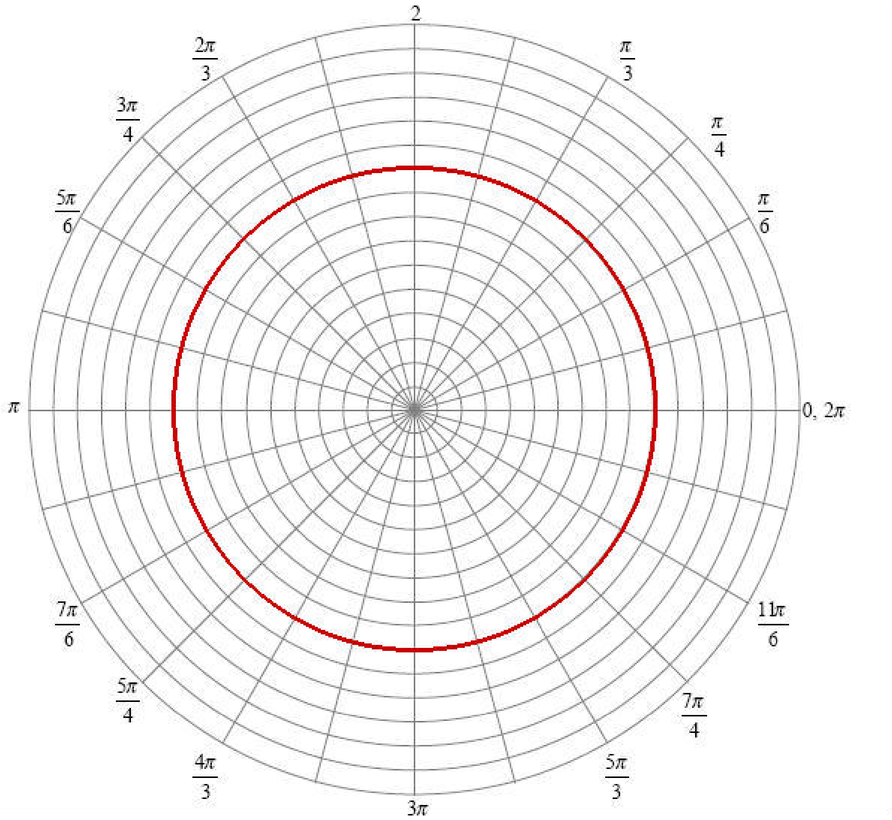




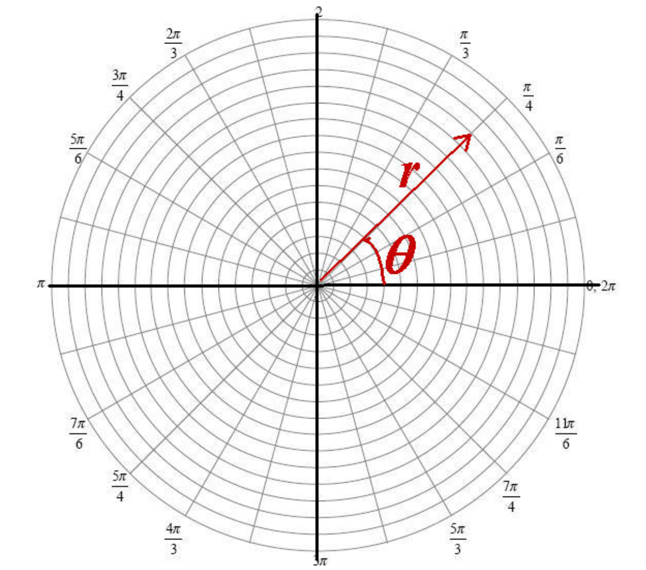




2
$\frac{2\pi}{3}$
$\frac{\pi}{3}$
$\frac{3\pi}{4}$
$\frac{\pi}{4}$
$\frac{5\pi}{6}$
$\frac{\pi}{6}$
π
0, 2π
$\frac{7\pi}{6}$
$\frac{11\pi}{6}$
$\frac{5\pi}{4}$
$\frac{7\pi}{4}$
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
3π

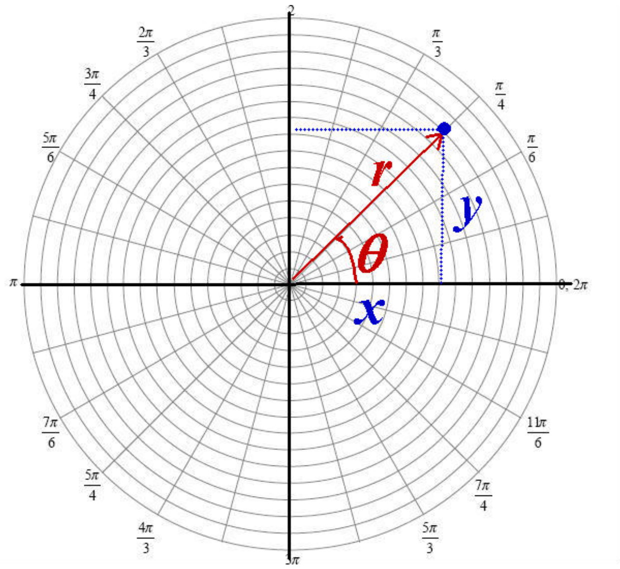

2
π/3
π/4
π/6
0, 2π
11π/6
7π/4
5π/3
3π
4π/3
5π/4
7π/6
π
5π/6
3π/4
2π/3

Specify a point with radius r and angle θ

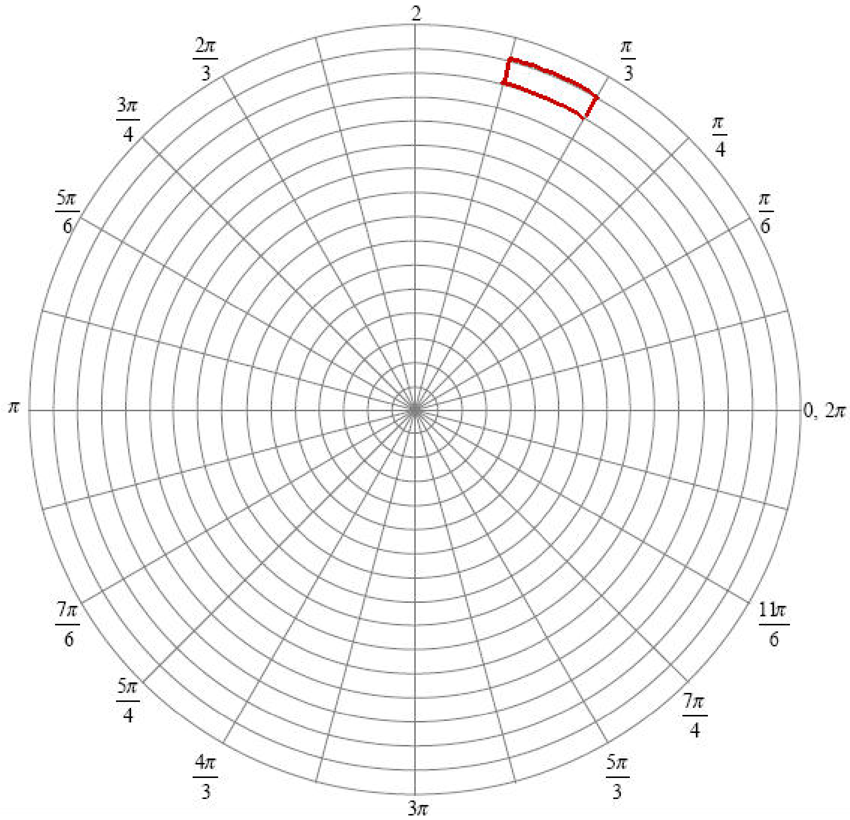


$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

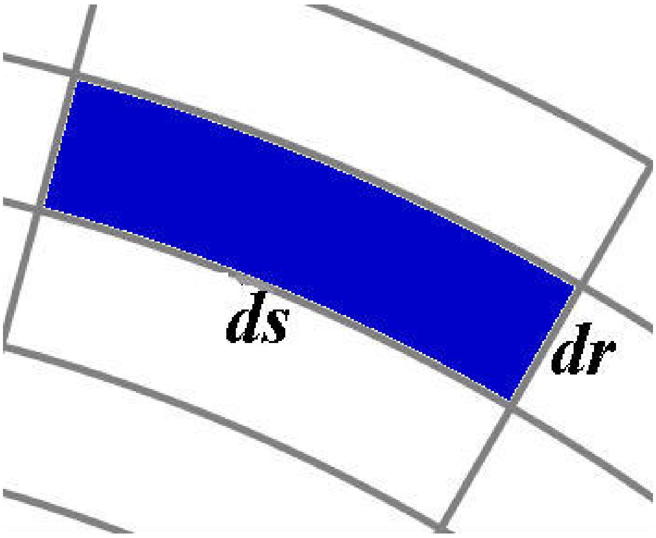


$$\iint_R f(x,\, y)\, dA = \iint_R f(r\cos\theta,\ r\sin\theta)\, dA$$

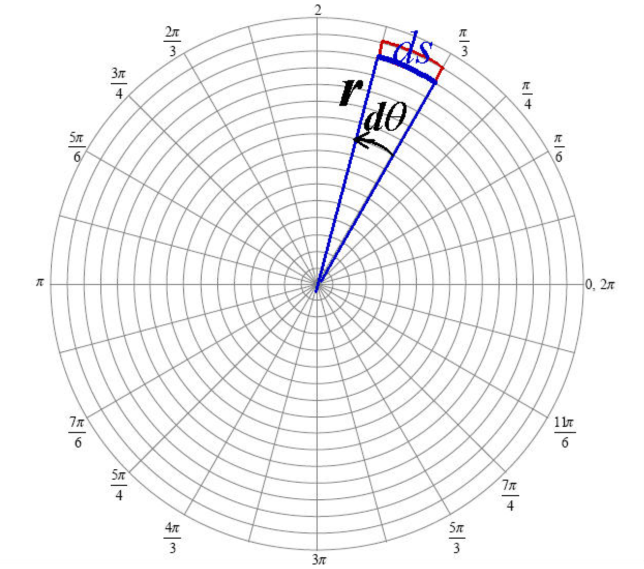


2
2π/3
π/3
3π/4
π/4
5π/6
π/6
π
0, 2π
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π

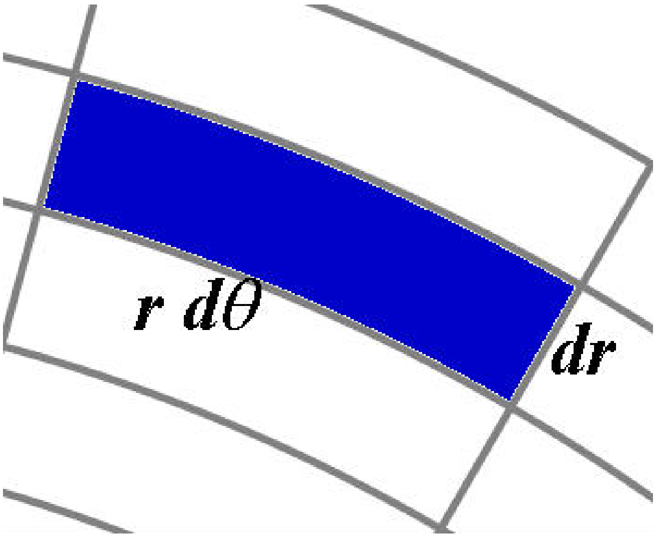
$$dA = ds\,dr$$



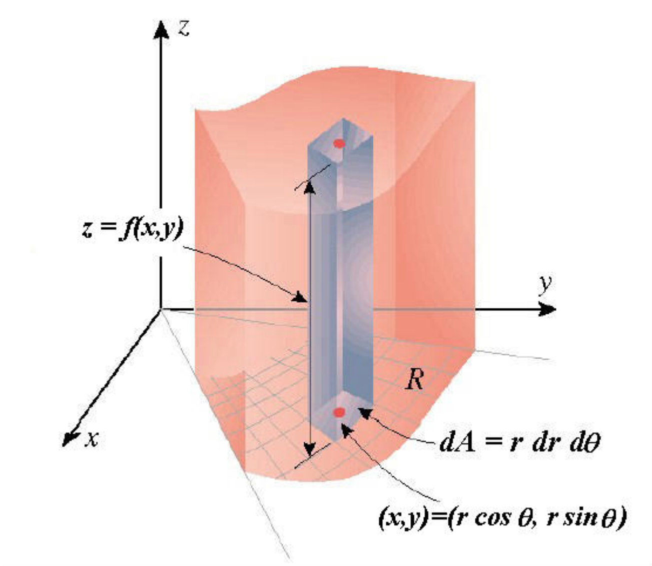
$$d\theta = \frac{ds}{r} \quad \text{so} \quad ds = r\,d\theta$$



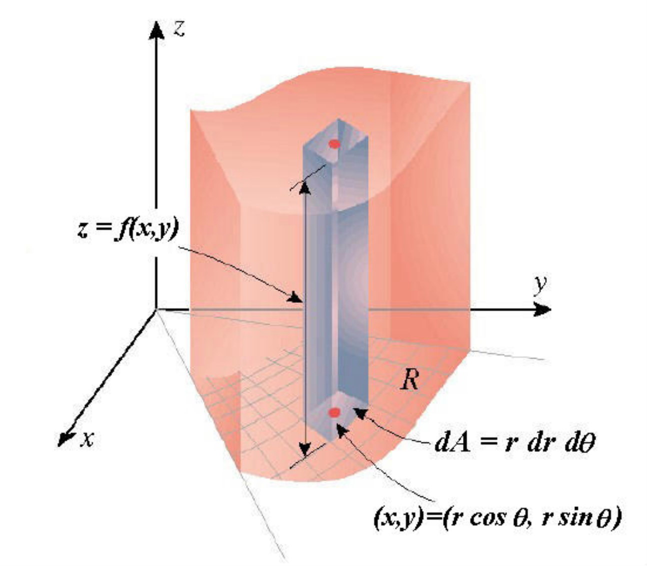
$$dA = ds\,dr = r\,d\theta\,dr$$



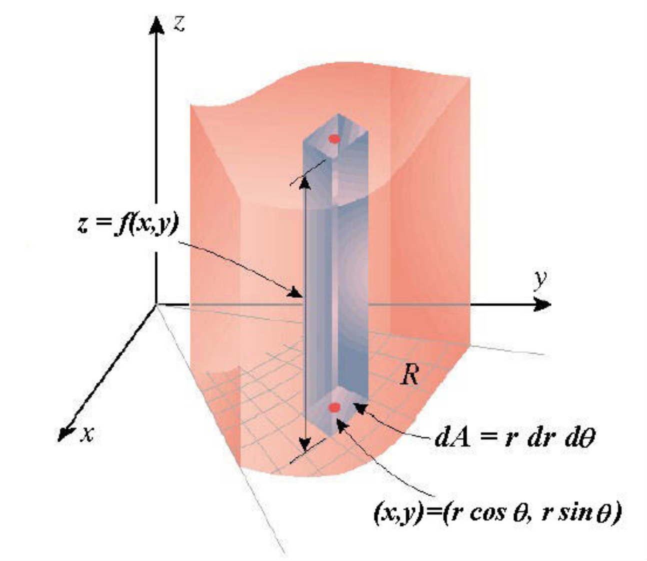
$$dV = f(x,\ y)\, dA = f(r\cos\theta,\, r\sin\theta)\, r\, d\theta\, dr$$



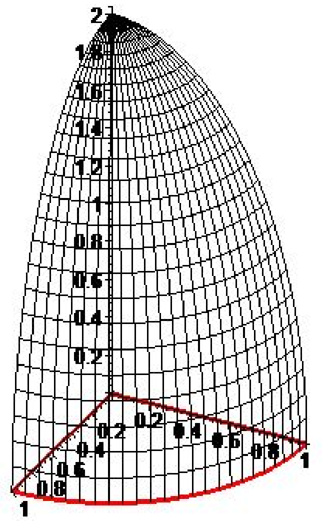
$$V = \iint_R f(r\cos\theta,\ r\sin\theta)\, r\, d\theta\, dr$$



$$V = \iint_R f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$



$$\iint_Q 2\sqrt{1-x^2-y^2}\,dA$$



$$x = r\cos\theta \qquad y = r\sin\theta$$

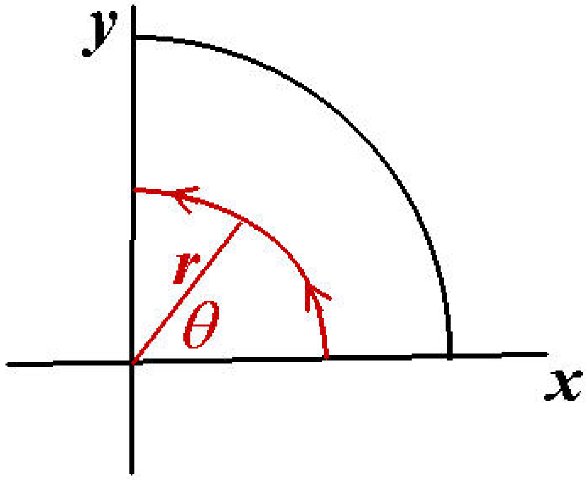
$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

Therefore,

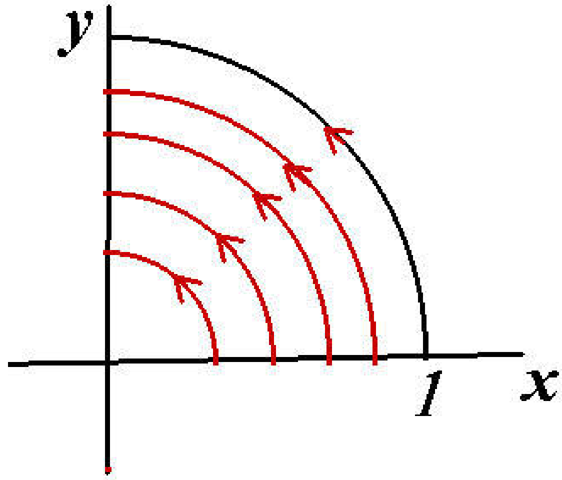
$$\sqrt{1 - x^2 - y^2} = \sqrt{1 - (x^2 + y^2)} = \sqrt{1 - r^2}$$

$$\iint_Q 2\sqrt{1 - x^2 - y^2}\, dA = \iint_Q 2\sqrt{1 - r^2}\; r\, d\theta\, dr$$

$$\iint_Q 2\sqrt{1-x^2-y^2}\,dA = \int\int_0^{\pi/2} 2\sqrt{1-r^2}\cdot r\,d\theta\,dr$$



$$\iint_Q 2\sqrt{1-x^2-y^2}\,dA = \int_0^1 \int_0^{\pi/2} 2\sqrt{1-r^2}\cdot r\,d\theta\,dr$$

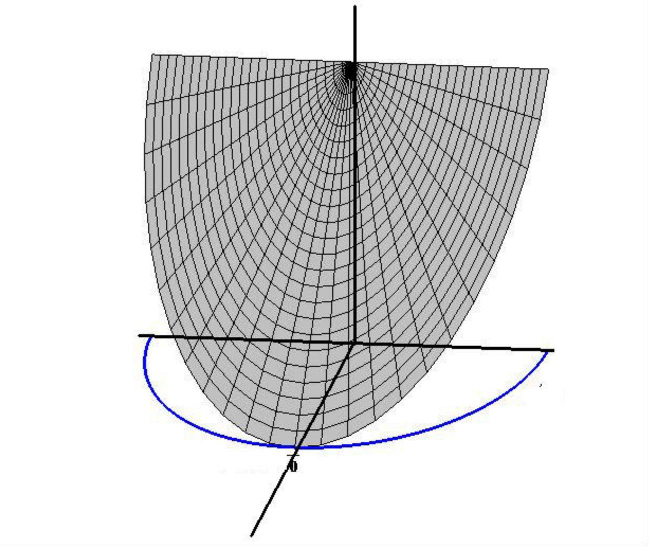


$$\begin{aligned}\iint_Q 2\sqrt{1-x^2-y^2}\,dA &= \int_0^1\int_0^{\pi/2} 2\sqrt{1-r^2}\cdot r\,d\theta\,dr\\ &= \int_0^1 \left[2\sqrt{1-r^2}\cdot\theta\cdot r\right]_{\theta=0}^{\pi/2} dr\end{aligned}$$

$$\begin{aligned}\iint_Q 2\sqrt{1-x^2-y^2}\,dA &= \int_0^1\int_0^{\pi/2} 2\sqrt{1-r^2}\cdot r\,d\theta\,dr\\ &= \int_0^1 \left[2\sqrt{1-r^2}\cdot\theta\cdot r\right]_{\theta=0}^{\pi/2} dr\\ &= \pi\int_0^1 \left(1-r^2\right)^{1/2}\, r\,dr\\ &= \frac{\pi}{3}\end{aligned}$$

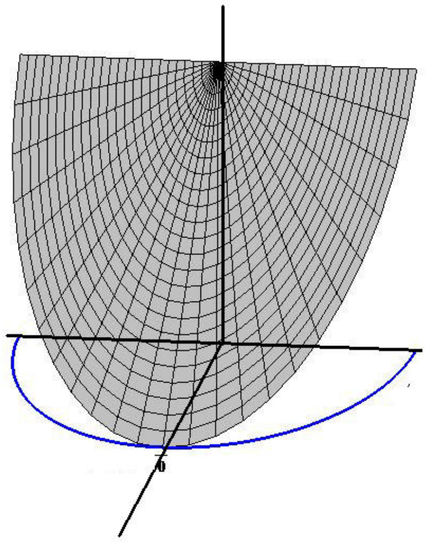
Let R be the semicircular region in the xy plane bounded by the y-axis and $x = \sqrt{1 - y^2}$

$$V = \iint_R (3 - 3x)\, dA$$

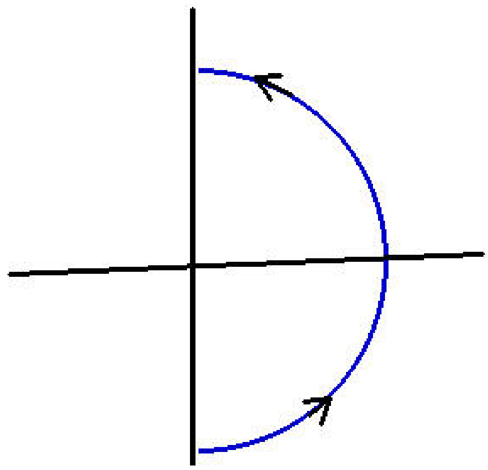


Let R be the semicircular region in the xy plane bounded by the y-axis and $x = \sqrt{1-y^2}$

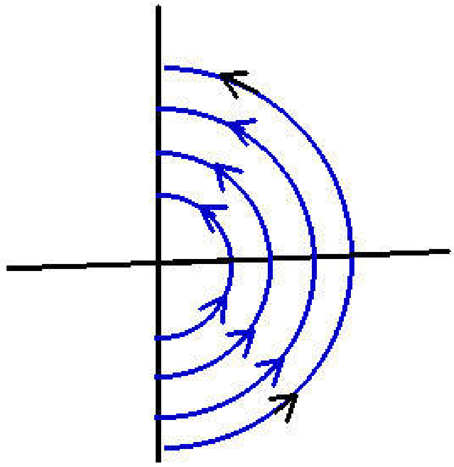
$$V = \iint_R (3 - 3x)\, dA = \iint (3 - 3r\cos\theta)\, r\, d\theta\, dr$$



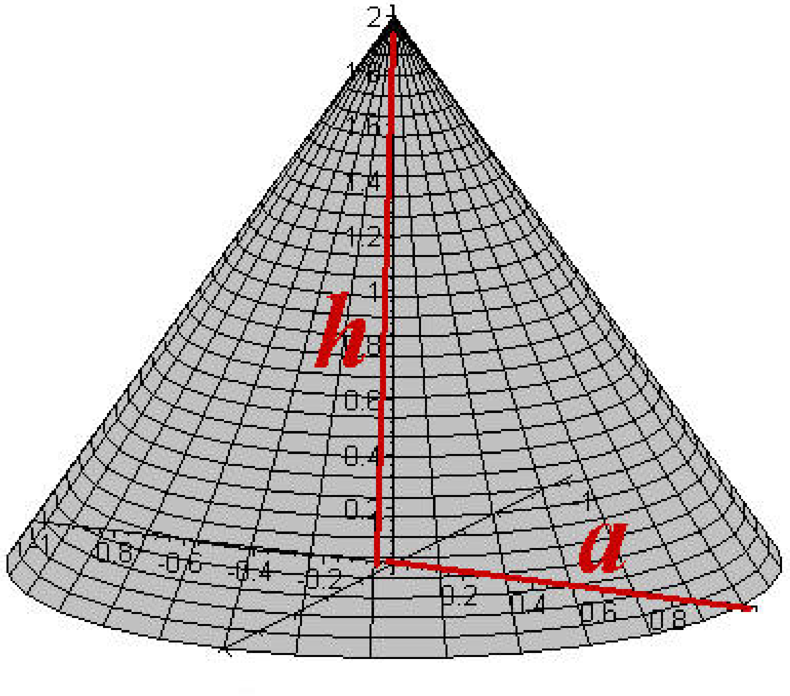
$$\int\int_{-\pi/2}^{\pi/2} (3 - 3r\cos\theta)\, r\, d\theta\, dr$$



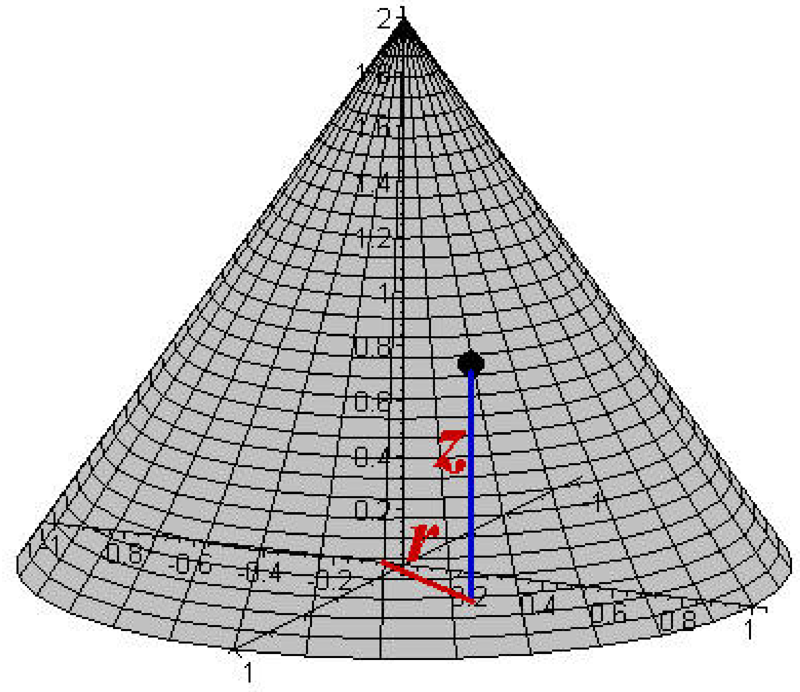
$$\int_0^1 \int_{-\pi/2}^{\pi/2} (3 - 3r\cos\theta)\, r\, d\theta\, dr$$

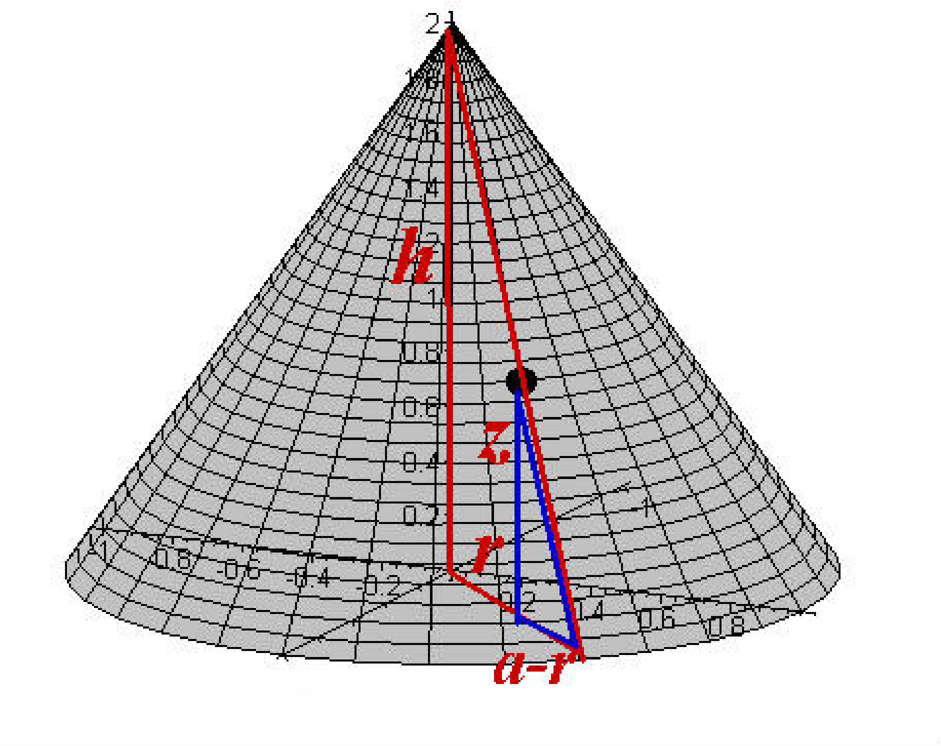


$$\begin{aligned}\int_0^1 \int_{-\pi/2}^{\pi/2} (3 - 3r\cos\theta)\, r \, d\theta \, dr &= \int_0^1 \left(3\pi r - 3r^2\right) \, dr \\ &= \frac{3\pi - 4}{2}\end{aligned}$$

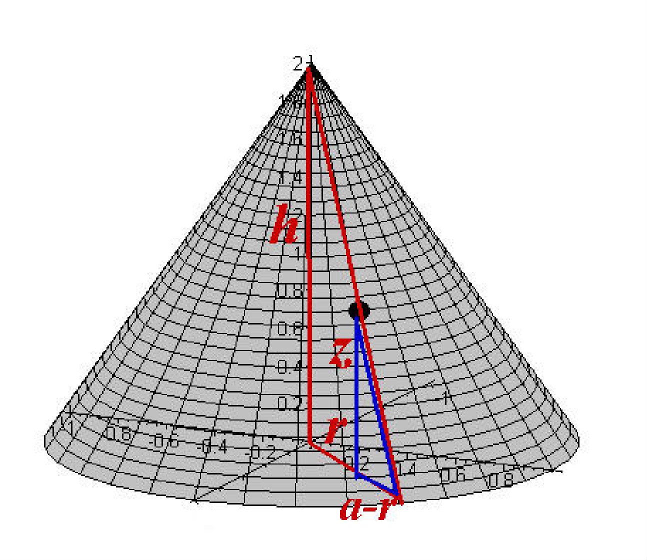


h
a


z
r


h
z
r
a-r

$$\frac{z}{a-r} = \frac{h}{a}$$



$$\frac{z}{a-r} = \frac{h}{a}$$

$$z = \frac{h}{a}(a-r) = \frac{h}{a}\left(a - \sqrt{x^2+y^2}\right)$$

$$
\begin{aligned}
V &= \iint_R \frac{h}{a}(a - r)\, dA \\
&= \iint_R \frac{h}{a}(a - r)\, r \, d\theta \, dr
\end{aligned}
$$

$$
\begin{aligned}
V &= \iint_R \frac{h}{a}(a-r)\,dA \\
&= \iint_R \frac{h}{a}(a-r)\,r\,d\theta\,dr \\
&= \int_0^a \int_0^{2\pi} \frac{h}{a}(a-r)\,r\,d\theta\,dr \\
&= \frac{h}{a}\int_0^a \int_0^{2\pi} (ar-r^2)\,d\theta\,dr
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{h}{a}\int_0^a \int_0^{2\pi} (ar - r^2)\, d\theta\, dr \\
&= \frac{h}{a}\int_0^a \left[(ar - r^2)\theta\right]_{\theta=0}^{2\pi} dr
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{h}{a}\int_0^a\int_0^{2\pi}(ar-r^2)\,d\theta\,dr\\
&= \frac{h}{a}\int_0^a \Big[(ar-r^2)\theta\Big]_{\theta=0}^{2\pi}\,dr\\
&= \frac{2\pi h}{a}\int_0^a (ar-r^2)\,dr\\
&= \frac{2\pi h}{a}\Big[\frac{1}{2}ar^2-\frac{1}{3}r^3\Big]_{r=0}^{a}\\
&= \frac{1}{3}\pi a^2 h
\end{aligned}
$$