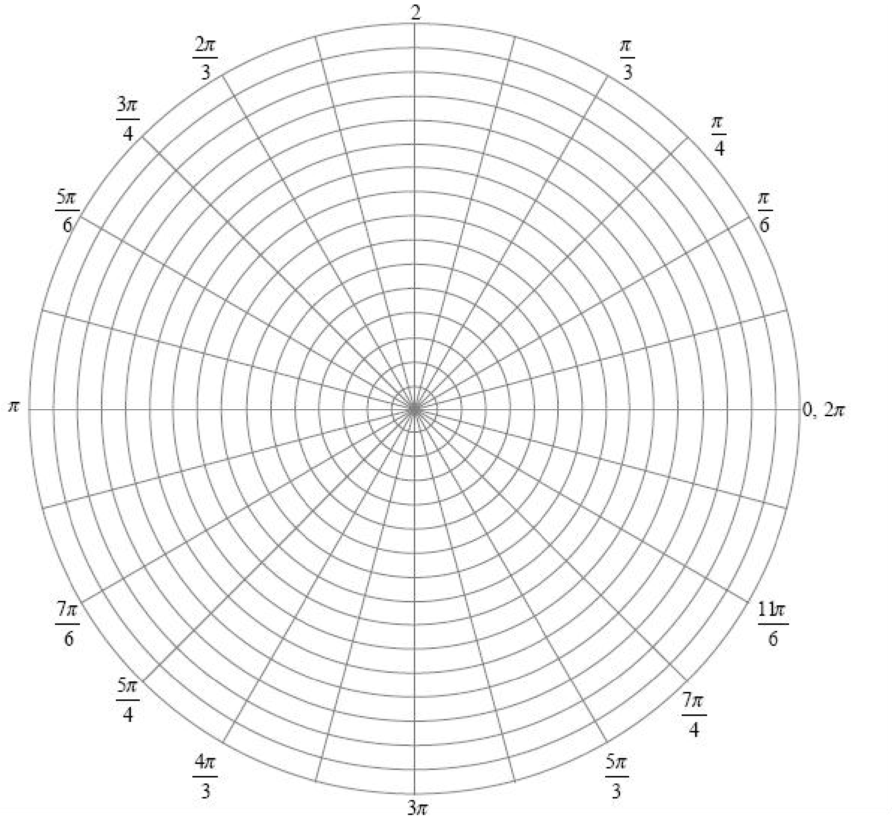
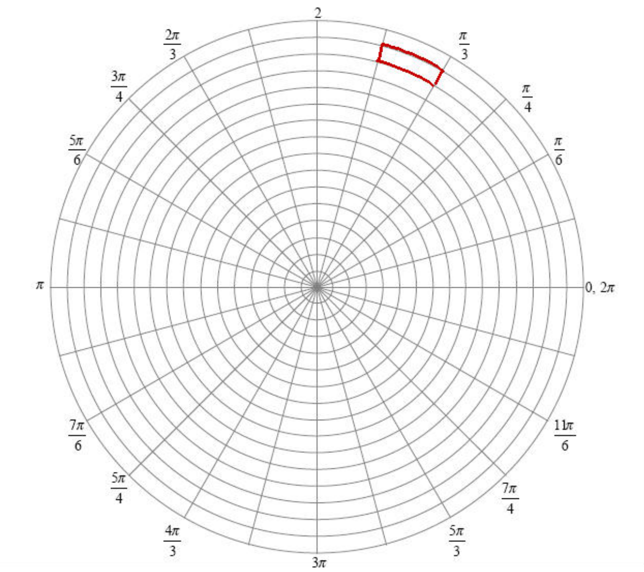
Polar Coordinates - Examples



$$dA = r\, d\theta\, dr$$



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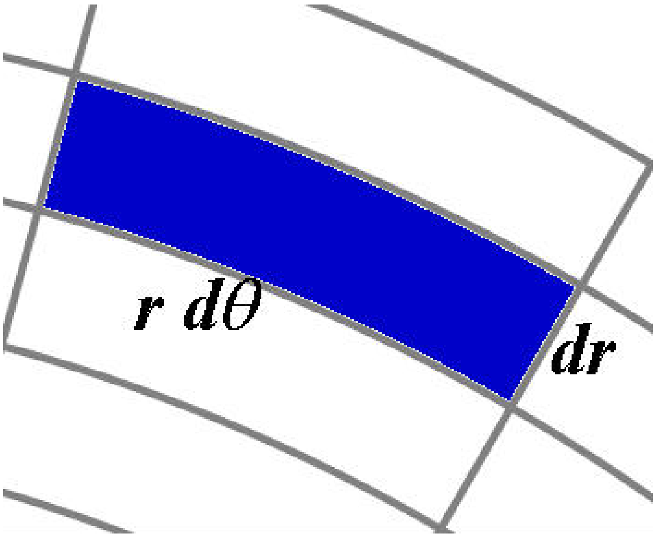
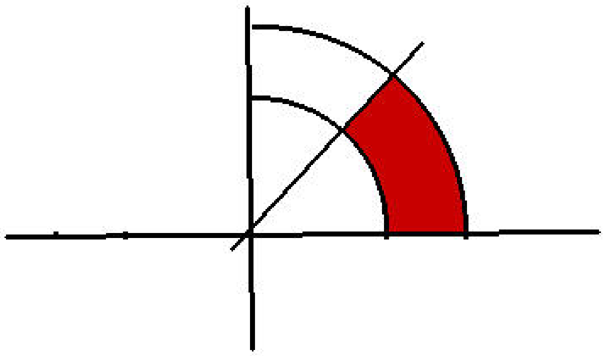


Let Q be the section in the xy plane bounded by:

$y = 0 \qquad y = x \qquad y = \sqrt{1 - x^2} \qquad y = \sqrt{4 - x^2}$



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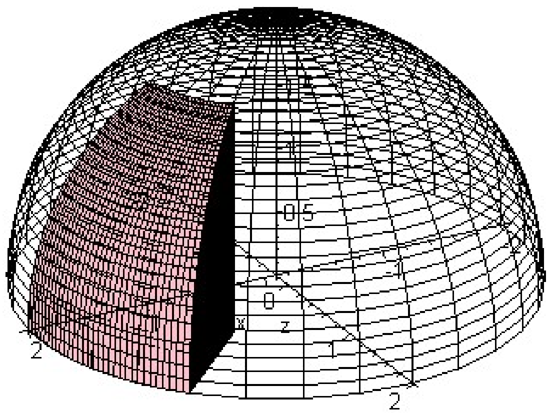
$$\iint_Q \sqrt{4 - x^2 - y^2}\, dA$$



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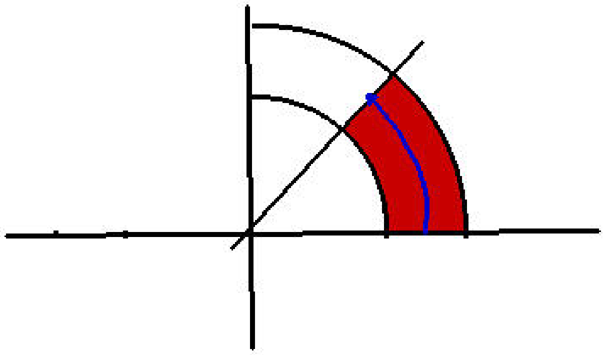
$$\iint_Q \sqrt{4-x^2-y^2}\, dA = \int_?^? \int_0^{\pi/4} \sqrt{4-r^2}\, r\, d\theta\, dr$$



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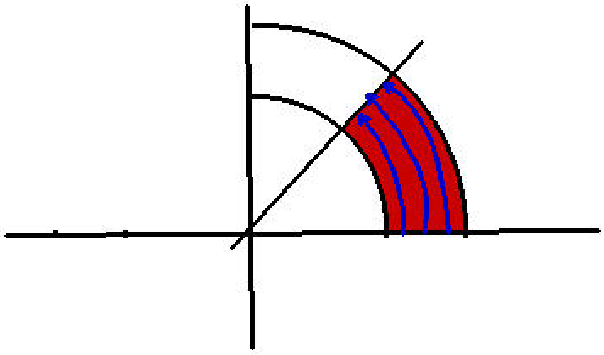

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$$= \int_1^2 \left[\sqrt{4 - r^2}\, r\, \theta \right]_0^{\pi/4} dr$$

$$= \int_1^2 \frac{\pi}{4} \left(4 - r^2\right)^{1/2} r\, dr$$

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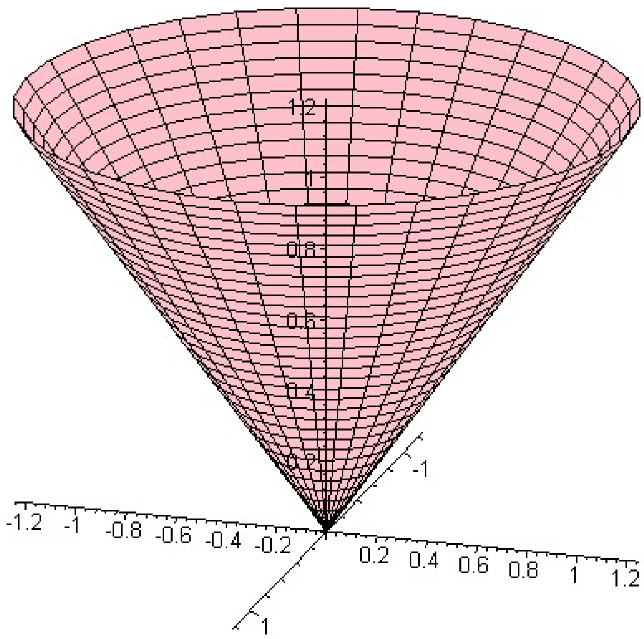
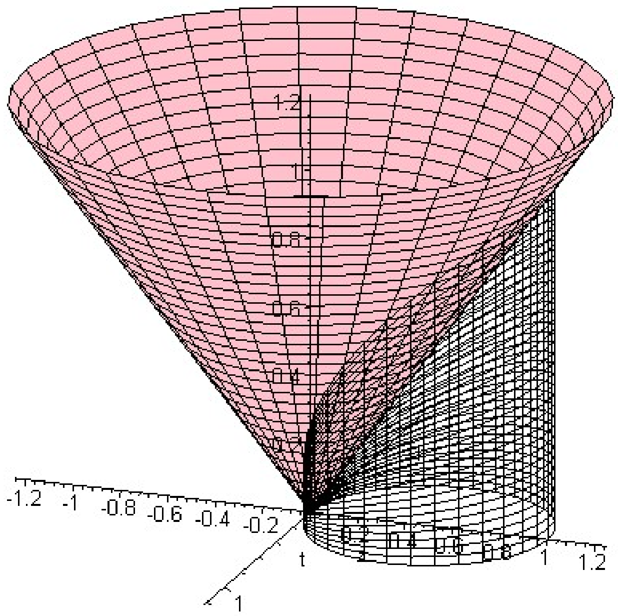
$$= \frac{\pi\sqrt{3}}{4}$$

$$z = \sqrt{x^2 + y^2}$$

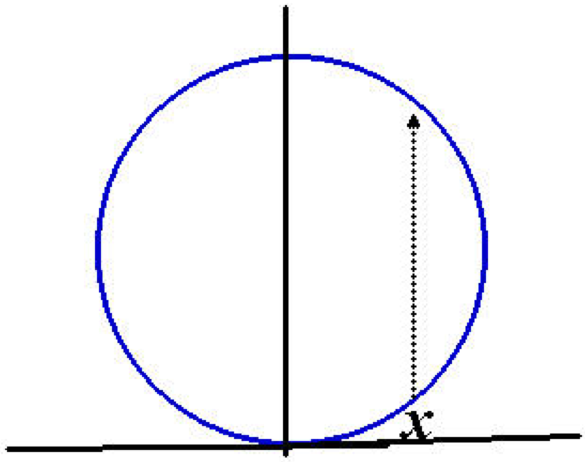


Let $\mathcal{D}$ be the disk in the xy plane inside of the circle $x^2 + (y-1)^2 = 1$. Calculate $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\, dA$



$$\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dA = \iint \sqrt{x^2 + y^2}\, dy\, dx$$

If $x^2 + (y-1)^2 = 1$ then $y = 1 \pm \sqrt{1 - x^2}$



$$x^2 + (y-1)^2 = 1$$

$$(r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1$$

$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$$

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$$r^2(\cos^2\theta + \sin^2\theta) - 2r\sin\theta = 0$$

$$r^2 - 2r\sin\theta = 0$$

$$r = 2\sin\theta$$

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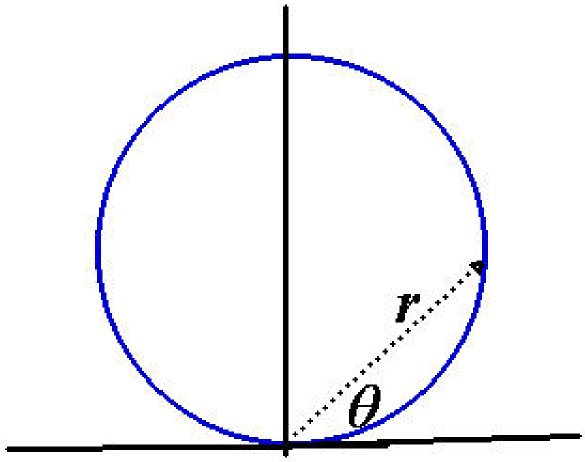
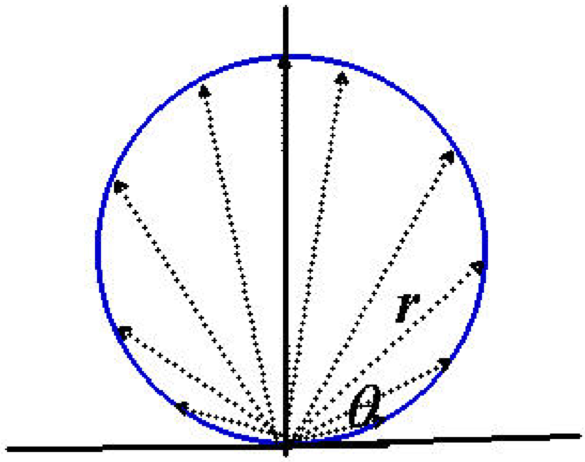


$$r = 2\sin\theta$$



$$\sqrt{x^2 + y^2} = \sqrt{r^2} = r$$

$$\begin{aligned}
\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dA &= \int_0^{\pi} \int_0^{2\sin\theta} r \cdot r\, dr\, d\theta \\
&= \int_0^{\pi} \int_0^{2\sin\theta} r^2\, dr\, d\theta
\end{aligned}$$

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\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dA &= \int_0^{\pi} \int_0^{2\sin\theta} r \cdot r \, dr \, d\theta \\
&= \int_0^{\pi} \int_0^{2\sin\theta} r^2 \, dr \, d\theta \\
&= \int_0^{\pi} \frac{8}{3} \sin^3\theta \, d\theta
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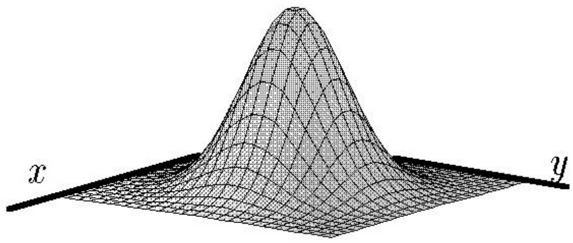
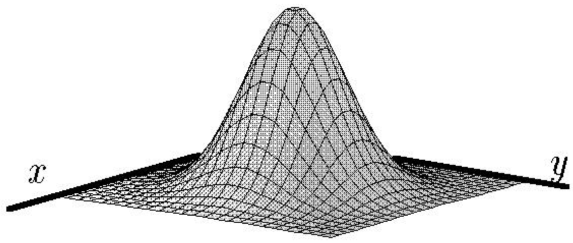
&= \int_0^{\pi} \int_0^{2\sin\theta} r^2 \, dr \, d\theta \\
&= \int_0^{\pi} \frac{8}{3} \sin^3\theta \, d\theta \\
&= \frac{8}{3} \int_0^{\pi} (1-\cos^2\theta) \sin\theta \, d\theta \\
&= \frac{32}{9}
\end{aligned}$$

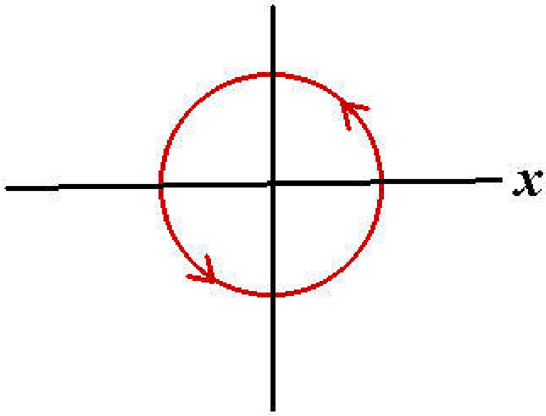
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\, dy\, dx$$



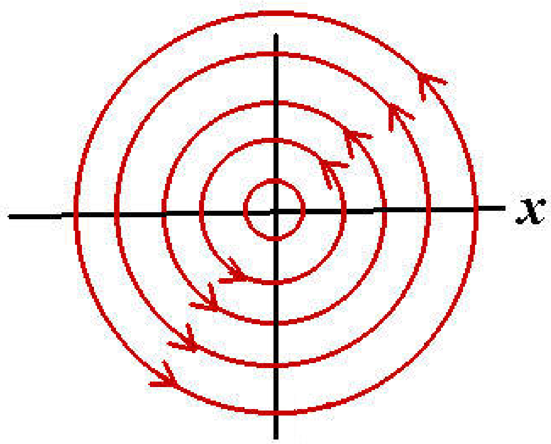
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\, dy\, dx = \int_{?}^{?}\int_{?}^{?} e^{-\frac{1}{2}r^2}\, r\, d\theta\, dr$$



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dy\,dx = \int_{?}^{?}\int_{0}^{2\pi} e^{-\frac{1}{2}r^2}\,r\,d\theta\,dr$$

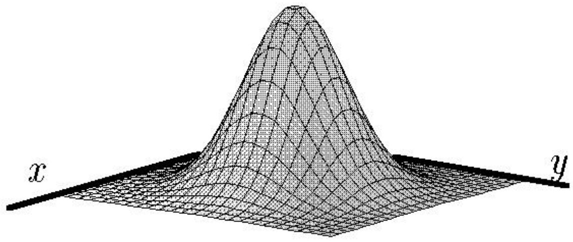


$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dy\,dx = \int_{0}^{\infty}\int_{0}^{2\pi} e^{-\frac{1}{2}r^2}\,r\,d\theta\,dr$$

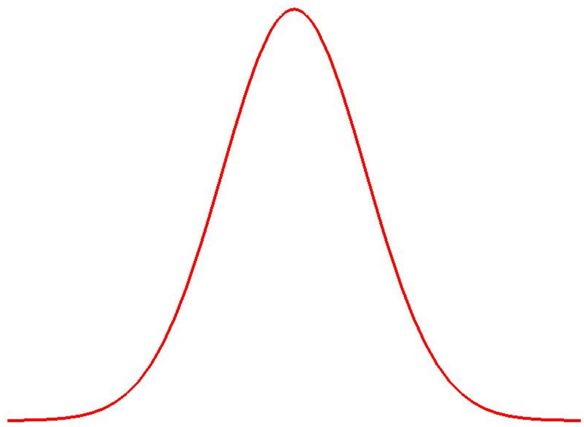


$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dy\,dx &= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{1}{2}r^2}\;r\,d\theta\,dr \\
&= \int_0^{\infty}\left[e^{-\frac{1}{2}r^2} r\theta\right]_{\theta=0}^{2\pi} dr \\
&= \int_0^{\infty} 2\pi e^{-\frac{1}{2}r^2}\;r\,dr \\
&= 2\pi
\end{aligned}
$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dy\,dx = 2\pi$$



$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx$$



$$\int_0^1 x^3 \, dx = \frac{1}{4}$$

$$\int_0^1 y^3 \, dy = \frac{1}{4}$$

$$\int_0^1 u^3 \, du = \frac{1}{4}$$

$$\vdots$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy$$

$$e^{-\frac{1}{2}(x^2+y^2)} = e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2}$$

So, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} \, dy \, dx$ is the same as:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} \, dy \, dx$$

Which is the product of two single integrals:

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx \right) \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \, dy \right)$$

$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy$ Therefore,

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy\right)$$

is the same as:

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)$$

which equals:

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)^2$$

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dy\,dx$$

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)^2 = 2\pi$$

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx\right)^2 = 2\pi$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$$

The Standard Normal Distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$