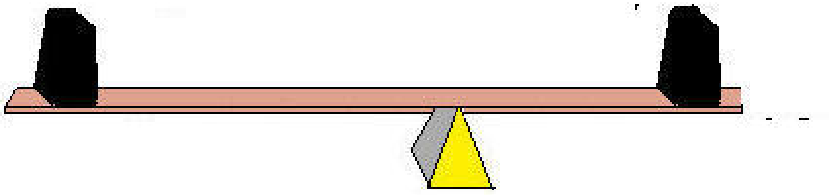
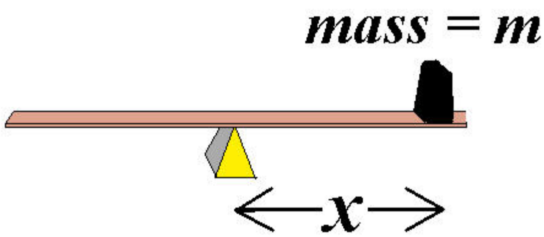
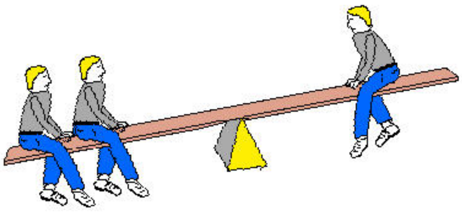
Center of Mass and Centroids



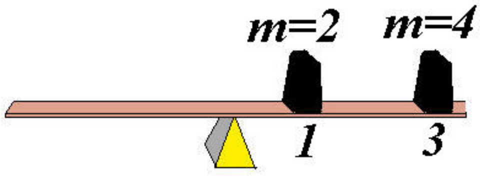
$$\textbf{Moment} = mx$$



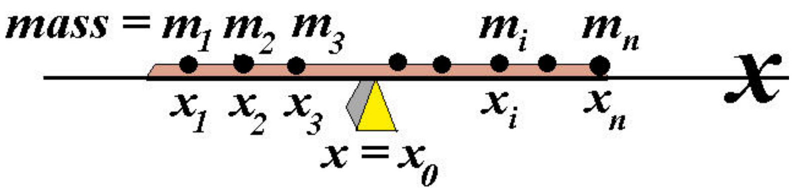
The moment measures the tendency to rotate around a particular point.



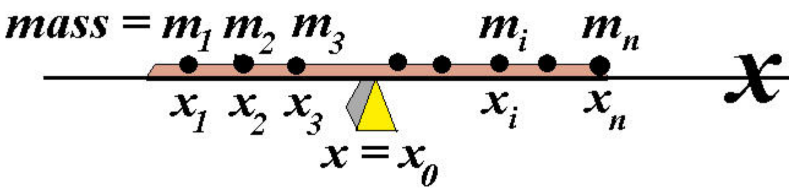
$$\text{Moment} = m_1 x_1 + m_2 x_2 = (2)(1) + (4)(3) = 14 \text{ kg m}$$



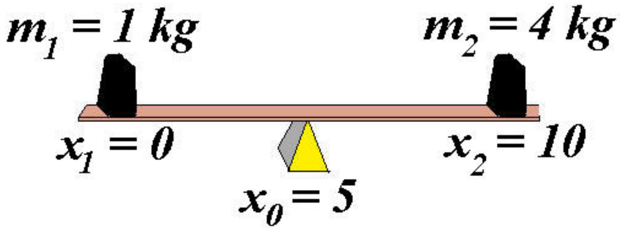
$$\mathbf{M}_{x_0} = m_1(x_1 - x_0) + m_2(x_2 - x_0) + \cdots + m_n(x_n - x_0)$$



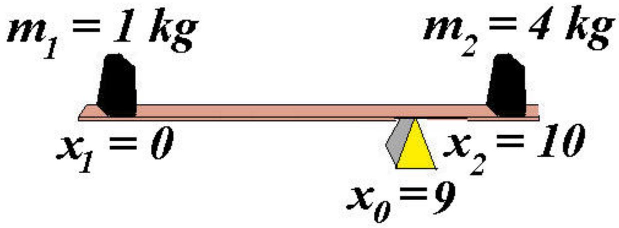
$$\mathbf{M}_{x_0} = \sum_{i=1}^{n} m_i(x_i - x_0)$$



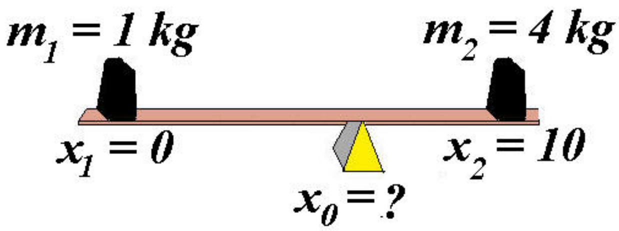
$$
\begin{aligned}
\mathbf{M}_{x_0} &= m_1(x_1 - x_0) + m_2(x_2 - x_0) \\
&= (1)(0-5) + (4)(10-5) \\
&= 15
\end{aligned}
$$



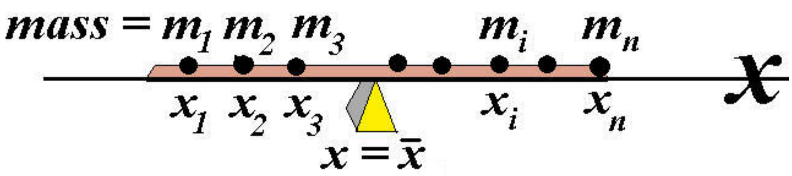
$$
\begin{aligned}
\mathbf{M}_{x_0} &= m_1(x_1 - x_0) + m_2(x_2 - x_0) \\
&= (1)(0-9) + (4)(10-9) \\
&= -5
\end{aligned}
$$



Where do we place the fulcrum so that the total moment is 0?



Let $x = \overline{x}$ be the value of x_0 that makes the total moment equal to 0. This is the center of mass.



The center of mass $\overline{x}$ has the property:

$$\mathbf{M}_{\overline{x}} = \sum_{i=1}^{n} m_i(x_i - \overline{x}) = 0$$

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} m_i (x_i - \overline{x}) \\
&= \sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i \overline{x} \\
&= \sum_{i=1}^{n} m_i x_i - \overline{x} \sum_{i=1}^{n} m_i
\end{aligned}
$$

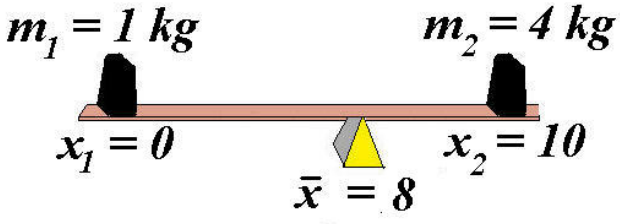
$$
\begin{aligned}
0 &= \sum_{i=1}^{n} m_i(x_i - \overline{x}) \\
&= \sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i \overline{x} \\
&= \sum_{i=1}^{n} m_i x_i - \overline{x} \sum_{i=1}^{n} m_i
\end{aligned}
$$

$$
\overline{x} \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i x_i
$$

Center of Mass

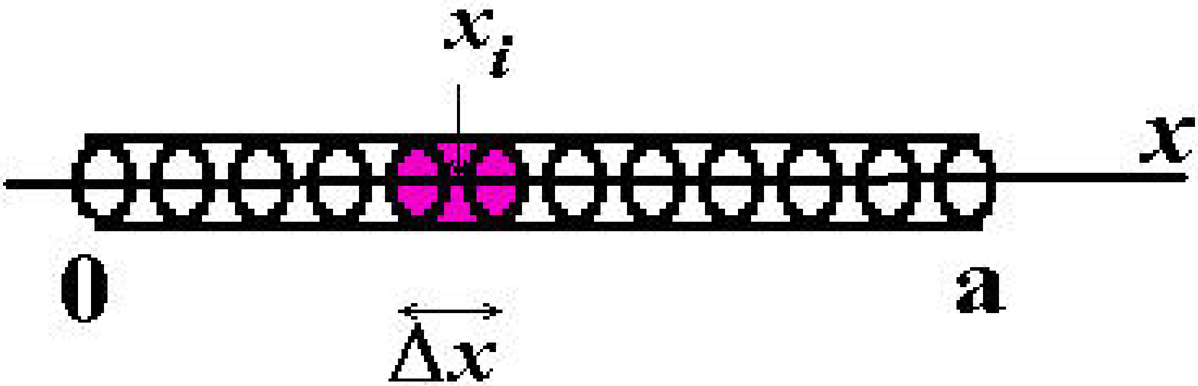
$$\overline{x} = \frac{\sum m_i x_i}{\sum m_i}$$

$$\overline{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(1)(0) + (4)(10)}{1 + 4} = 8$$

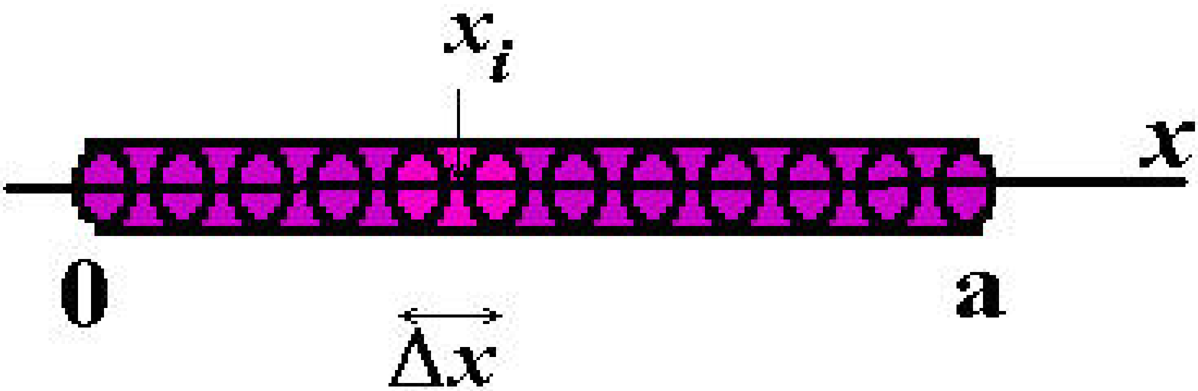


Let $\delta(x)$ be the density (kg/m).

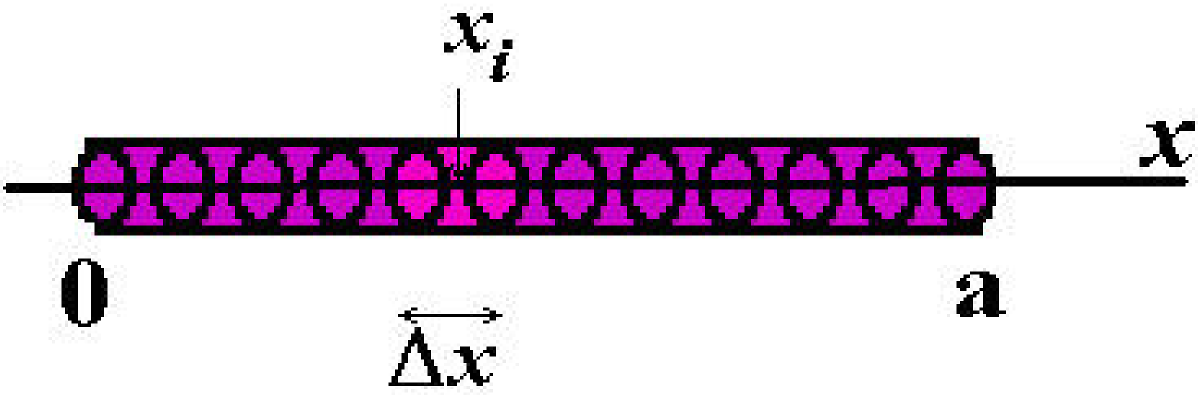
The amount of mass in the i^{th} section is approximately $\delta(x_i)\,\Delta x$



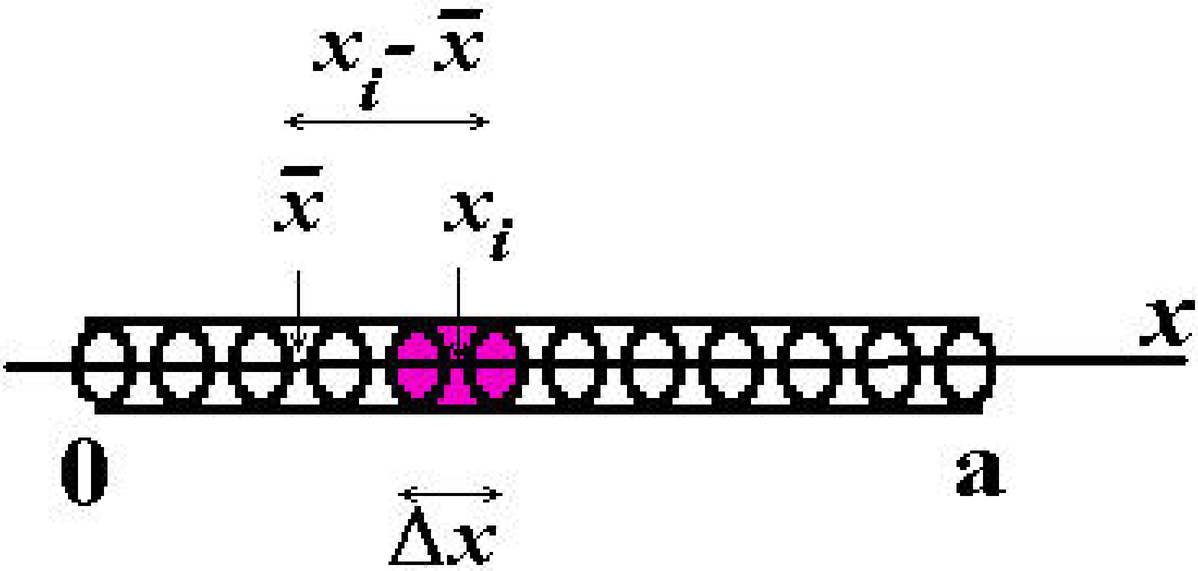
$$\text{Approximate total mass} = \sum_{i=1}^{n} \delta(x_i)\,\Delta x$$



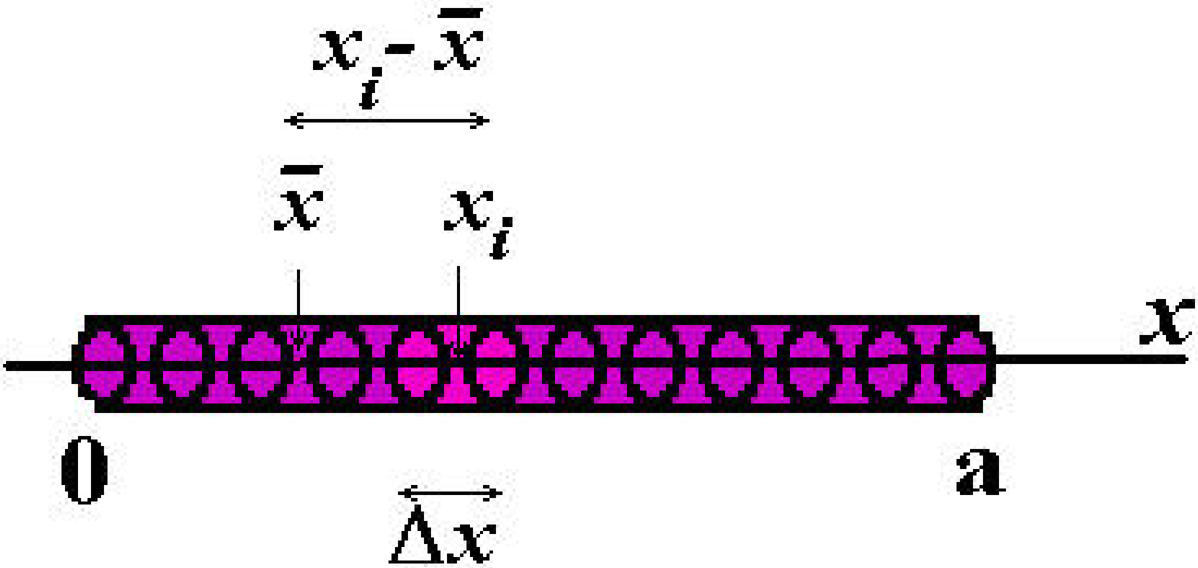
$$\text{Total mass} = \lim_{n\to\infty} \sum_{i=1}^{n} \delta(x_i)\,\Delta x = \int_0^a \delta(x)\,dx$$



$$\text{Moment of One Section} = (x_i - \overline{x})\delta(x_i)\,\Delta x$$



$$\text{Total Moment} = \int_0^a (x - \overline{x})\delta(x)\,dx$$



$$0 = \int_0^a (x - \overline{x})\delta(x)\,dx$$

$$0 = \int_0^a x\,\delta(x)\,dx - \int_0^a \overline{x}\,\delta(x)\,dx$$

$$0 = \int_0^a x\,\delta(x)\,dx - \overline{x}\int_0^a \delta(x)\,dx$$

$$0 = \int_0^a x\,\delta(x)\,dx - \overline{x}\int_0^a \delta(x)\,dx$$

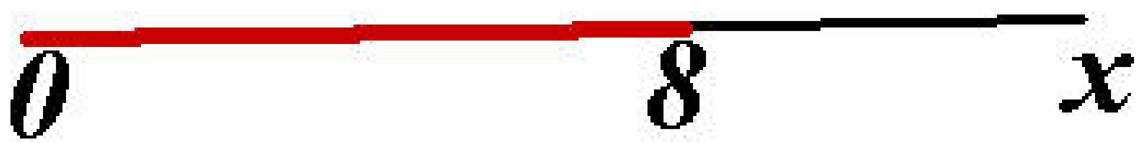
$$\overline{x}\int_0^a \delta(x)\,dx = \int_0^a x\,\delta(x)\,dx$$

$$0 = \int_0^a x\,\delta(x)\,dx - \overline{x}\int_0^a \delta(x)\,dx$$

$$\overline{x}\int_0^a \delta(x)\,dx = \int_0^a x\,\delta(x)\,dx$$

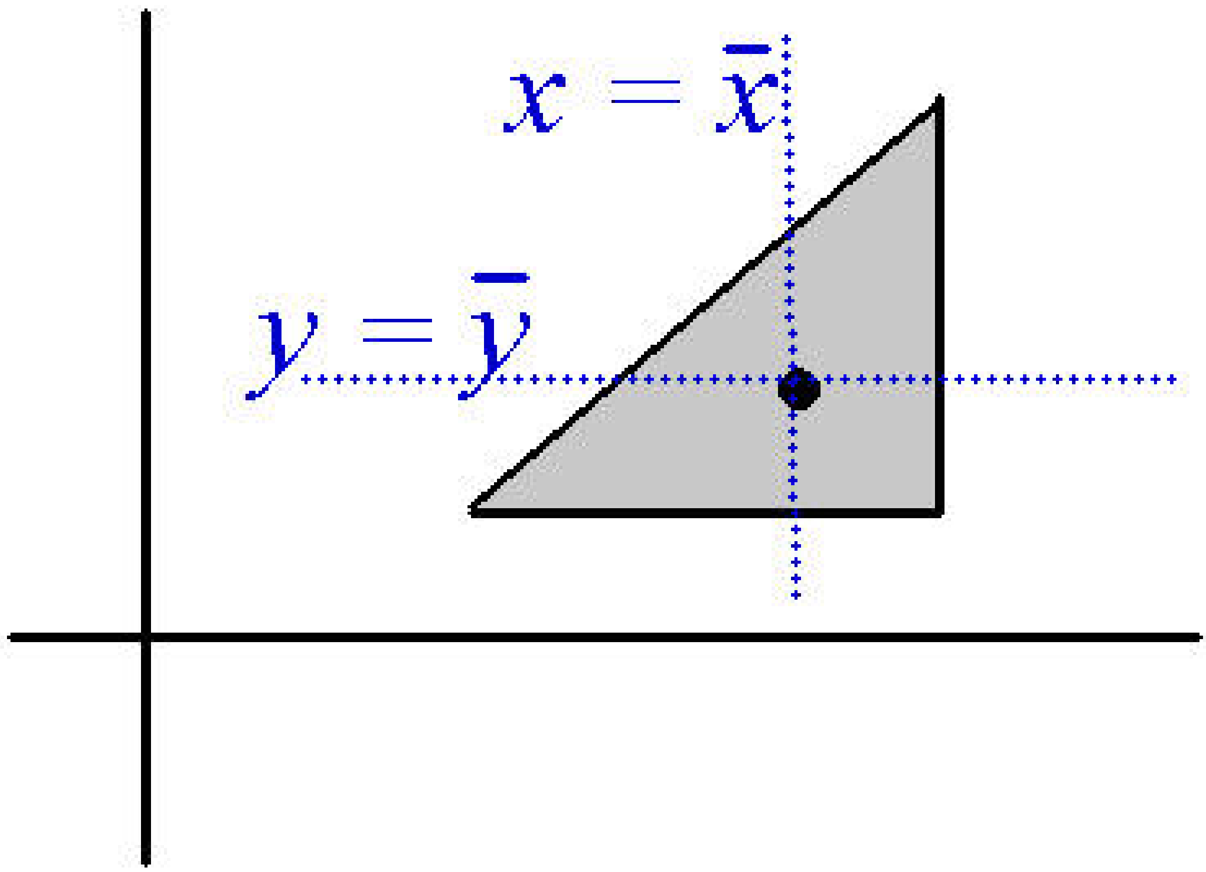
$$\overline{x} = \frac{\int_0^a x\,\delta(x)\,dx}{\int_0^a \delta(x)\,dx}$$

Let $\delta(x) = x^2$ for $0 \leq x \leq 8$



$$\overline{x} = \frac{\int_0^a x\,\delta(x)\,dx}{\int_0^a \delta(x)\,dx} = \frac{\int_0^8 x \cdot x^2\,dx}{\int_0^8 x^2\,dx}$$

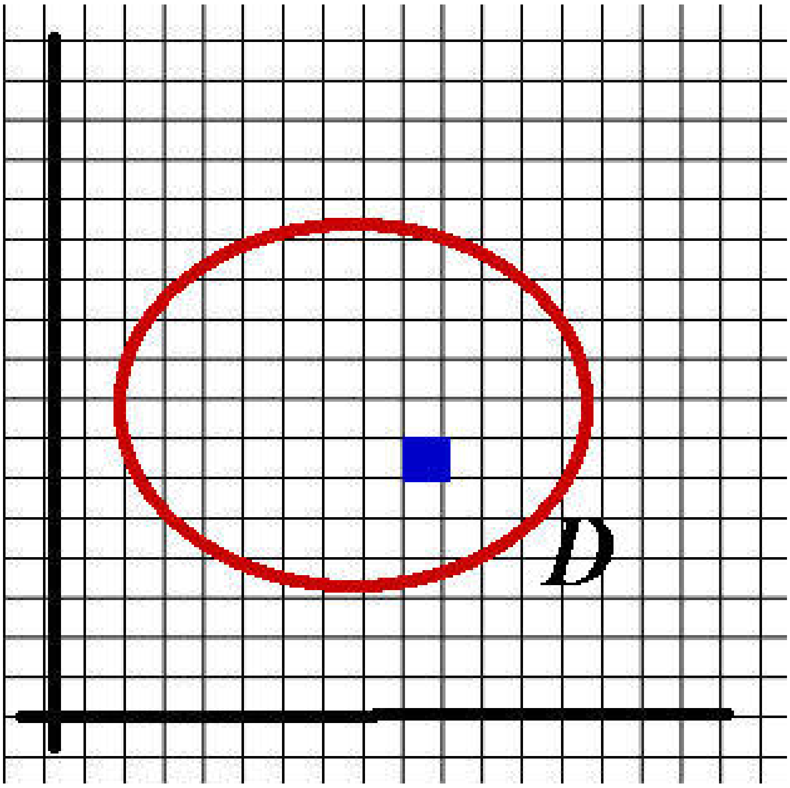
$$
\begin{aligned}
\overline{x} &= \frac{\int_0^a x\,\delta(x)\,dx}{\int_0^a \delta(x)\,dx} = \frac{\int_0^8 x \cdot x^2\,dx}{\int_0^8 x^2\,dx} \\
&= \frac{\left[\frac{1}{4}x^4\right]_0^8}{\left[\frac{1}{3}x^3\right]_0^8} \\
&= 6
\end{aligned}
$$



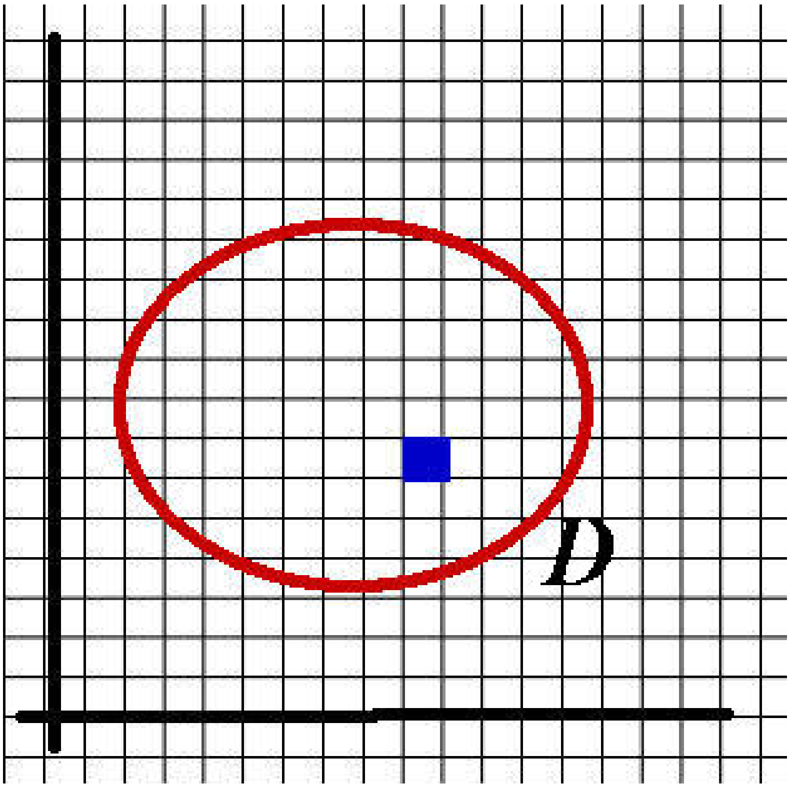
$x = \bar{x}$
$y = \bar{y}$

Let $\delta(x, y)$ be density (kg/m^2)

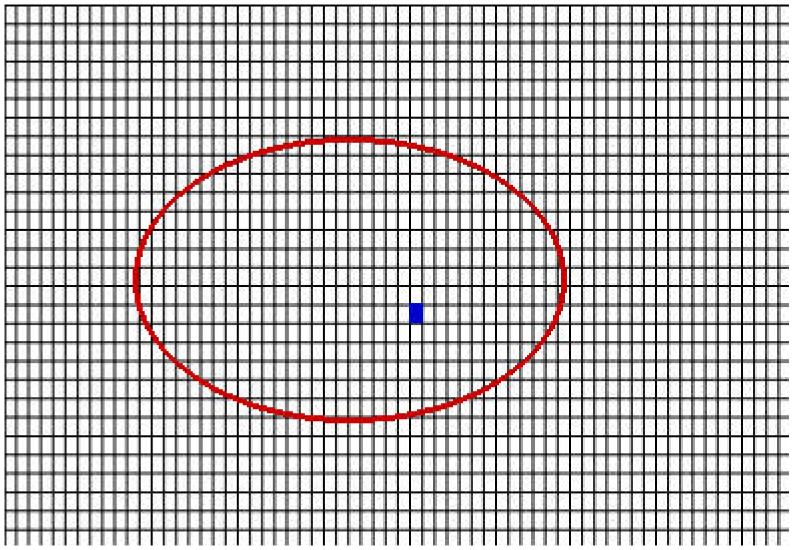
$\delta(x, y)\, dA$ is the mass of one section.



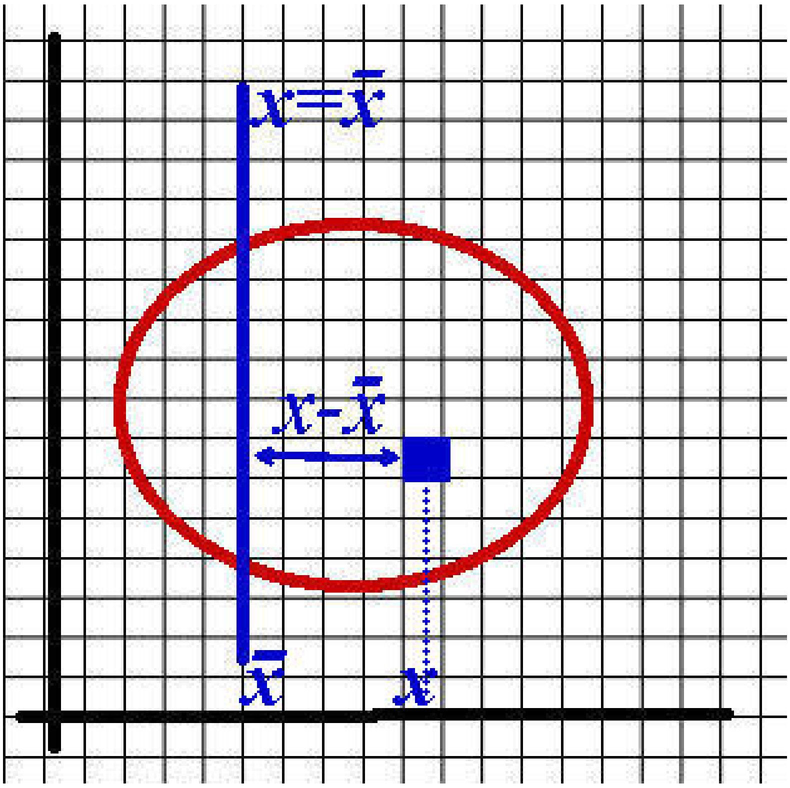
Let $\delta(x, y)$ be density (kg/m^2)
Mass in Region D is $\iint_D \delta(x, y)\, dA$



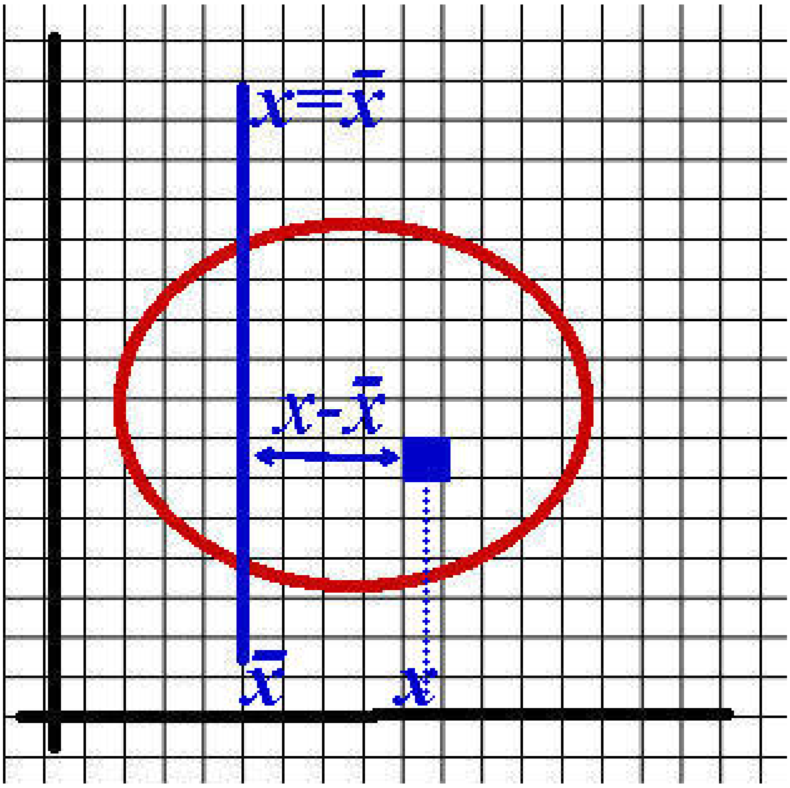
$$\int\int_D \delta(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \delta(x_i, y_j)\,\Delta A$$



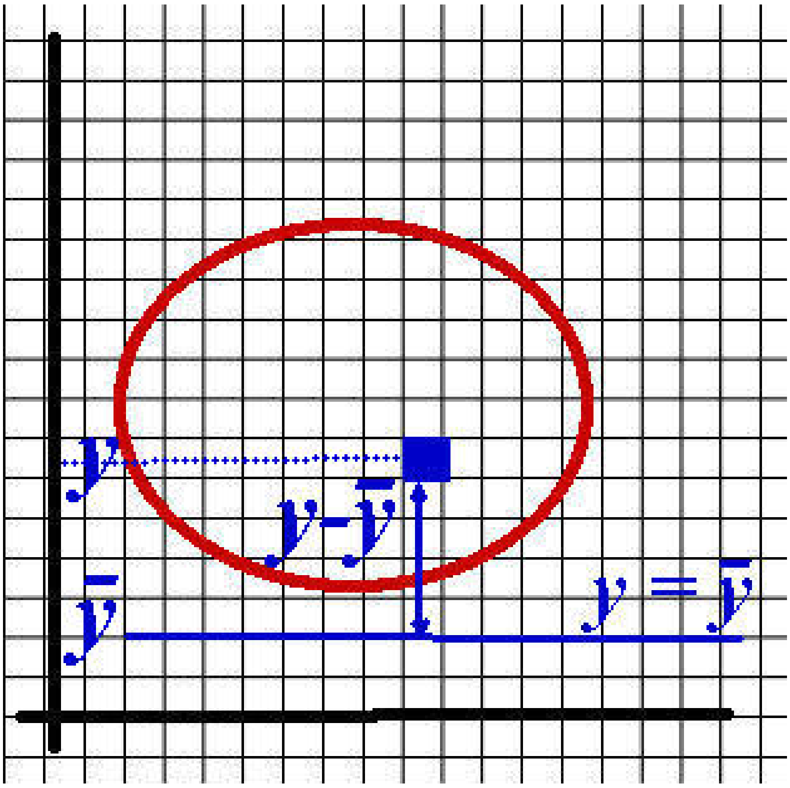
Moment of one section with respect to line $x = \overline{x}$ is $(x - \overline{x})\delta(x, y)\, dA$



$$M_{\overline{x}} = \iint_D (x - \overline{x})\delta(x, y)\, dA$$



$$M_{\overline{y}} = \iint_D (y - \overline{y})\delta(x, y)\, dA$$



$$M_{\overline{x}} = \iint_D (x - \overline{x})\delta(x, y)\, dA$$

$$0 = \iint_D x\,\delta(x, y)\, dA - \iint_D \overline{x}\delta(x, y)\, dA$$

$$M_{\overline{x}} = \iint_D (x - \overline{x})\delta(x, y)\, dA$$

$$0 = \iint_D x\,\delta(x, y)\, dA - \overline{x} \iint_D \delta(x, y)\, dA$$

$$M_{\overline{x}} = \iint_D (x - \overline{x})\delta(x, y)\, dA$$

$$0 = \iint_D x\,\delta(x, y)\, dA - \overline{x} \iint_D \delta(x, y)\, dA$$

$$\overline{x} \iint_D \delta(x, y)\, dA = \iint_D x\,\delta(x, y)\, dA$$

$$M_{\overline{x}} = \iint_D (x - \overline{x})\delta(x, y)\, dA$$

$$0 = \iint_D x\, \delta(x, y)\, dA - \overline{x} \iint_D \delta(x, y)\, dA$$

$$\overline{x} \iint_D \delta(x, y)\, dA = \iint_D x\, \delta(x, y)\, dA$$

$$\overline{x} = \frac{\iint_D x\, \delta(x, y)\, dA}{\iint_D \delta(x, y)\, dA}$$

$$M_{\overline{y}} = \iint_D (y - \overline{y})\delta(x, y)\, dA$$

Similarly, if $M_{\overline{y}} = 0$ then:

$$\overline{y} = \frac{\iint_D y\, \delta(x, y)\, dA}{\iint_D \delta(x, y)\, dA}$$

Center of mass in one dimension

$$\overline{x} = \frac{\int_0^a x\,\delta(x)\,dx}{\int_0^a \delta(x)\,dx}$$

Center of mass in two dimensions is $(\overline{x},\ \overline{y})$ where:

$$\overline{x} = \frac{\iint_D x\,\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA} \qquad \overline{y} = \frac{\iint_D y\,\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA}$$

Center of mass in one dimension

$$\overline{x} = \frac{\int_0^a x\,\delta(x)\,dx}{\int_0^a \delta(x)\,dx}$$

Center of mass in two dimensions is $(\overline{x},\ \overline{y})$ where:

$$\overline{x} = \frac{\iint_D x\,\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA} \qquad \overline{y} = \frac{\iint_D y\,\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA}$$

Definition: If δ is a constant, then the center of mass is called a *centroid*.

If δ is a constant, the formulas simplify

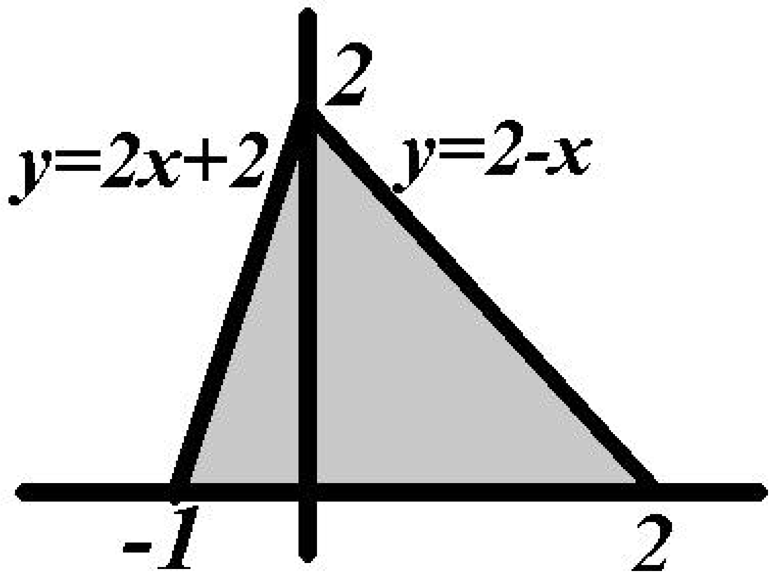
$$\begin{aligned}\overline{x} &= \frac{\iint_D x\,\delta\,dA}{\iint_D \delta\,dA} \\ &= \frac{\delta \iint_D x\ dA}{\delta \iint_D\ dA} \\ &= \frac{\iint_D x\ dA}{\iint_D\ dA}\end{aligned}$$

Note: $\iint_D 1\,dA$ is the area of the region D

The coordinates of the centroid are given by:

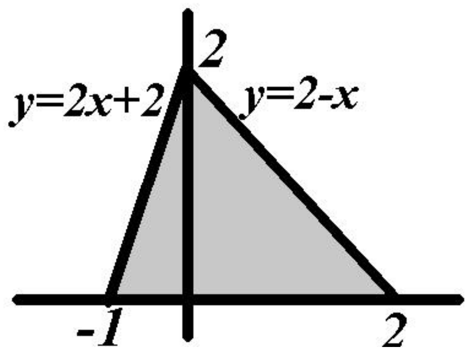
$$\overline{x} = \frac{1}{\text{Area}(D)} \iint_D x \, dA \qquad \overline{y} = \frac{1}{\text{Area}(D)} \iint_D y \, dA$$

Find the coordinates of the centroid of the triangle with vertices $(-1,\ 0)$, $(0,\ 2)$ and $(2,\ 0)$.



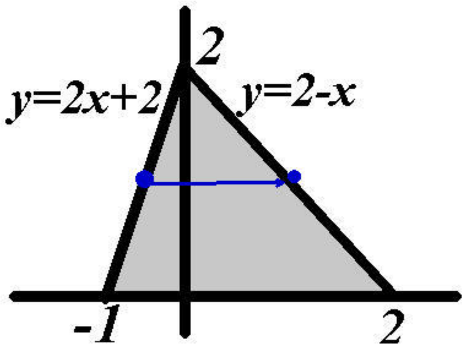
The area of this region is $\frac{1}{2}$(base)(height) $= 3$.

$$\overline{x} = \frac{1}{\text{Area}(D)} \iint_D x\, dA = \frac{1}{3} \iint_D x\, dx\, dy$$



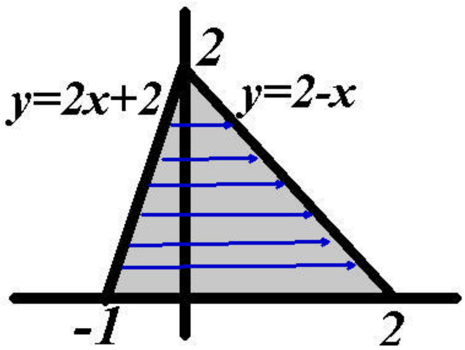
The area of this region is $\frac{1}{2}$(base)(height) $= 3$.

$$\overline{x} = \frac{1}{\text{Area}(D)} \iint_D x\, dA = \frac{1}{3} \int_{?}^{?} \int_{(y-2)/2}^{2-y} x\, dx\, dy$$



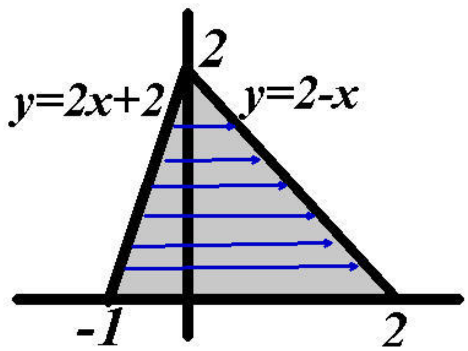
The area of this region is $\frac{1}{2}$(base)(height) $= 3$.

$$\overline{x} = \frac{1}{\text{Area}(D)} \iint_D x\, dA = \frac{1}{3} \int_0^2 \int_{(y-2)/2}^{2-y} x\, dx\, dy$$



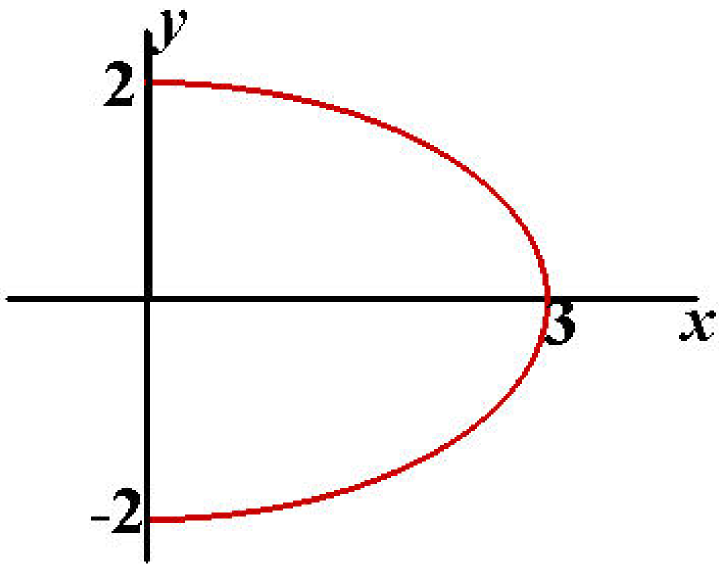
The area of this region is $\frac{1}{2}$(base)(height) $= 3$.

$$\overline{x} = \frac{1}{\text{Area}(D)} \iint_D x\,dA = \frac{1}{3}\int_0^2 \int_{(y-2)/2}^{2-y} x\,dx\,dy = \frac{1}{3}$$

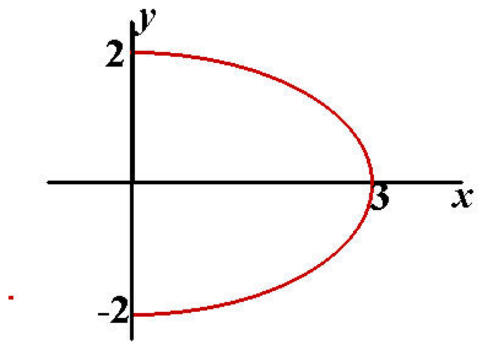


$$
\begin{aligned}
\overline{y} &= \frac{1}{3}\int_0^2 \int_{(y-2)/2}^{2-y} y\,dx\,dy \\
&= \frac{1}{3}\int_0^2 \Big[xy\Big]_{x=(y-2)/2}^{2-y} dy \\
&= \frac{1}{3}\int_0^2 \left(3y - \frac{3}{2}y^2\right)\,dy \\
&= \frac{2}{3}
\end{aligned}
$$

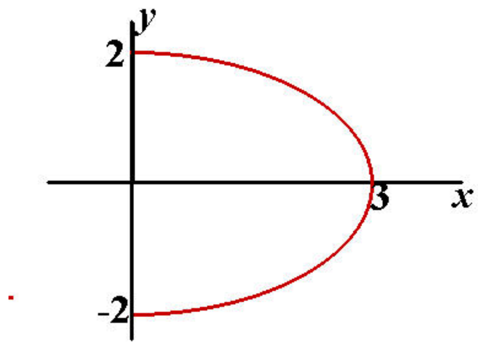
Find the centroid of the region bounded by the y-axis and the curve $x = \frac{3}{2}\sqrt{4 - y^2}$



$$
\begin{aligned}
\text{Area}(D) &= \int_{-2}^{2}\int_{0}^{\frac{3}{2}\sqrt{4-y^2}} dx\,dy \\
&= \int_{-2}^{2} \frac{3}{2}\sqrt{4-y^2}\,dy \\
&= 3\pi
\end{aligned}
$$



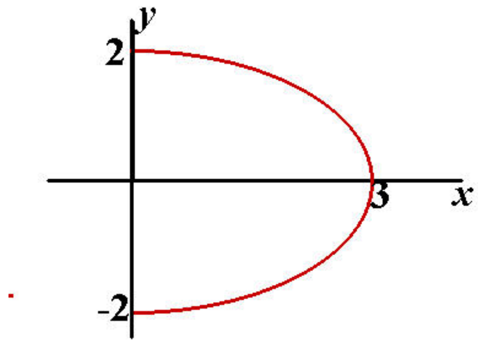
$$\overline{x} = \frac{1}{\text{Area}(D)} \iint_D x\, dA$$
$$= \frac{1}{3\pi} \int_{-2}^{2} \int_{0}^{\frac{3}{2}\sqrt{4-y^2}} x\, dx\, dy$$



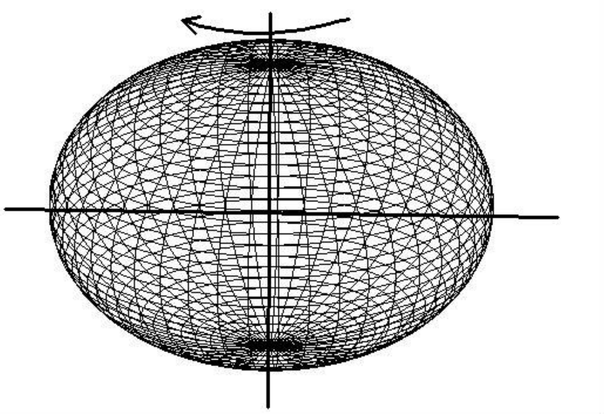
$$
\begin{aligned}
\overline{x} &= \frac{1}{\text{Area}(D)} \iint_D x \, dA \\
&= \frac{1}{3\pi} \int_{-2}^{2} \int_{0}^{\frac{3}{2}\sqrt{4-y^2}} x \, dx \, dy \\
&= \frac{1}{3\pi} \int_{-2}^{2} \left[\frac{1}{2} x^2 \right]_{x=0}^{\frac{3}{2}\sqrt{4-y^2}} dy \\
&= \frac{1}{3\pi} \int_{-2}^{2} \frac{9}{8} \left(4 - y^2\right) \, dy
\end{aligned}
$$

$$
\begin{aligned}
\overline{x} &= \frac{1}{\text{Area}(D)} \iint_D x \, dA \\
&= \frac{1}{3\pi} \int_{-2}^{2} \int_{0}^{\frac{3}{2}\sqrt{4-y^2}} x \, dx \, dy \\
&= \frac{1}{3\pi} \int_{-2}^{2} \left[\frac{1}{2} x^2 \right]_{x=0}^{\frac{3}{2}\sqrt{4-y^2}} dy \\
&= \frac{1}{3\pi} \int_{-2}^{2} \frac{9}{8} \left(4 - y^2 \right) \, dy \\
&= \frac{4}{\pi}
\end{aligned}
$$

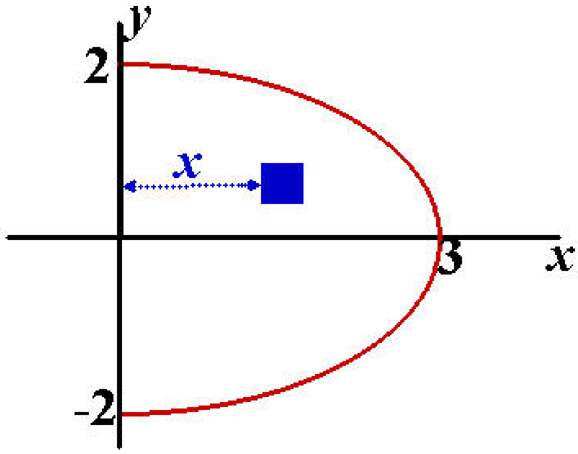
$$\overline{y} = \frac{1}{3\pi} \iint_D y \, dA = 0$$



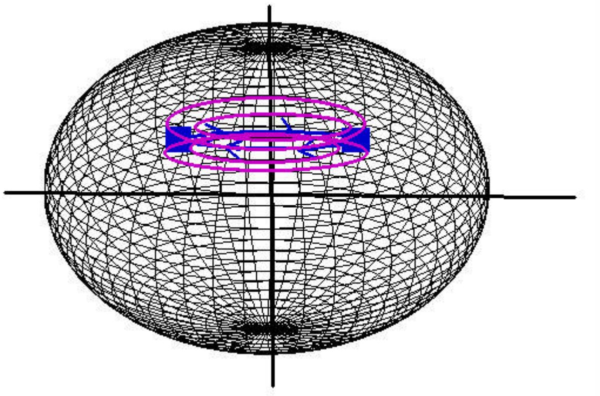
Revolve region D around the y-axis.



Look at the volume generated by one section of area dA that is a distance of x from the y-axis.

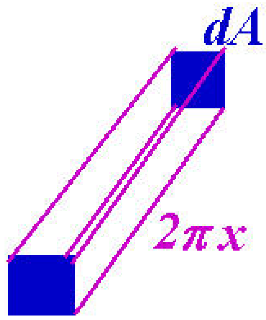


Look at the volume generated by one section of area dA that is a distance of x from the y-axis.

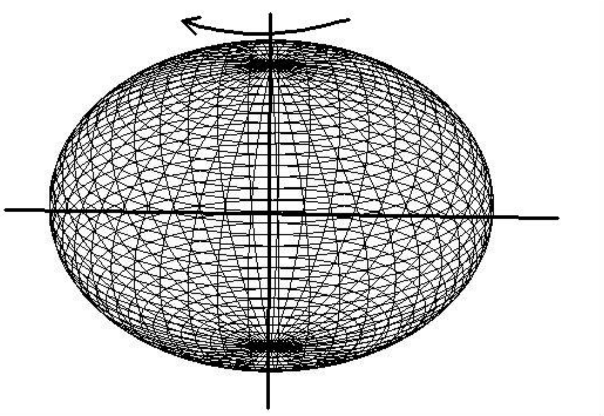


Straighten out the solid generated by one section.

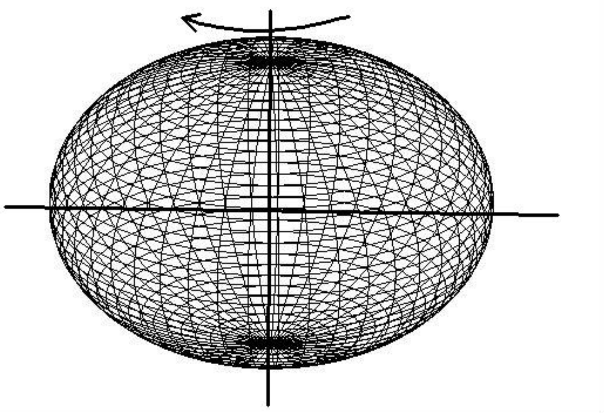
$$dV = 2\pi x \, dA$$



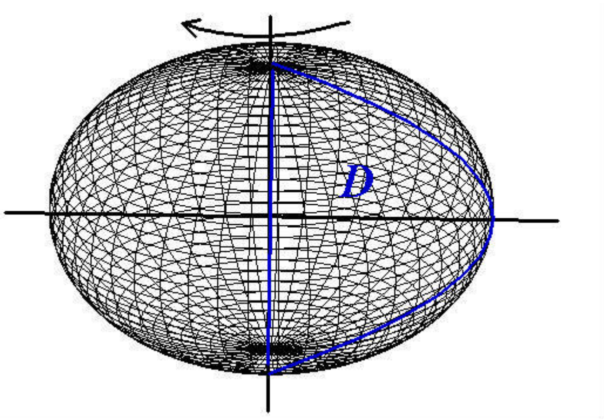
$$V = \iint_D 2\pi x \, dA$$



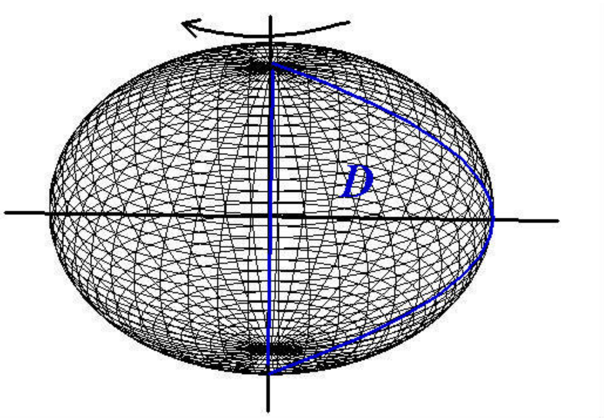
$$V = 2\pi \iint_D x \, dA$$



$$V = 2\pi \cdot \left(\frac{\iint_D x \, dA}{\text{Area}(D)} \right) \text{Area}(D)$$



$$V = 2\pi \overline{x} \text{Area}(D)$$



$$
\begin{aligned}
V &= 2\pi\overline{x}\text{Area}(D) \\
&= 2\pi \cdot \frac{4}{\pi} \cdot 3\pi \\
&= 24\pi
\end{aligned}
$$