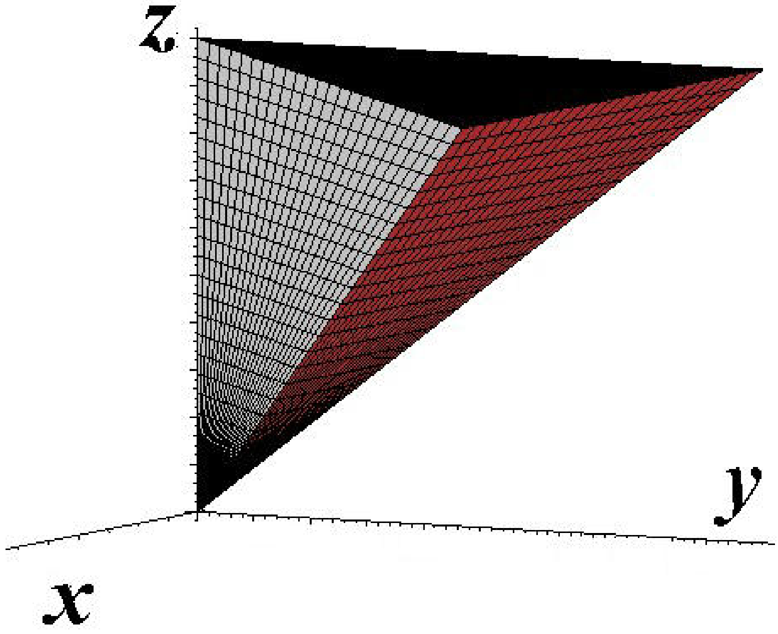
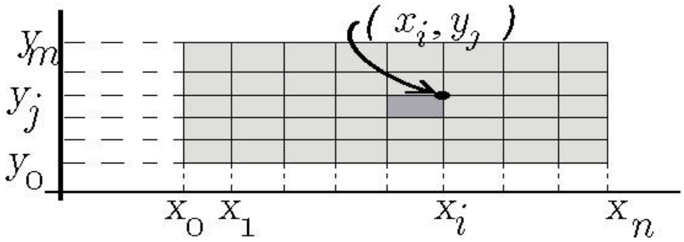
Introduction to Triple Integrals



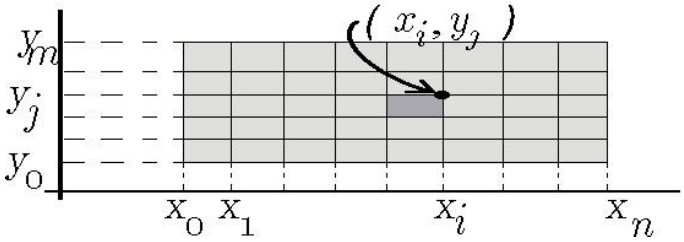
Let $f(x, y)$ be the mass density at (x, y) in kg/m^2
The mass of one section of the grid is:

$$f(x_i,\ y_j)\,\Delta x\,\Delta y$$



The total approximate mass in region R is:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i,\ y_j)\, \Delta x\, \Delta y$$

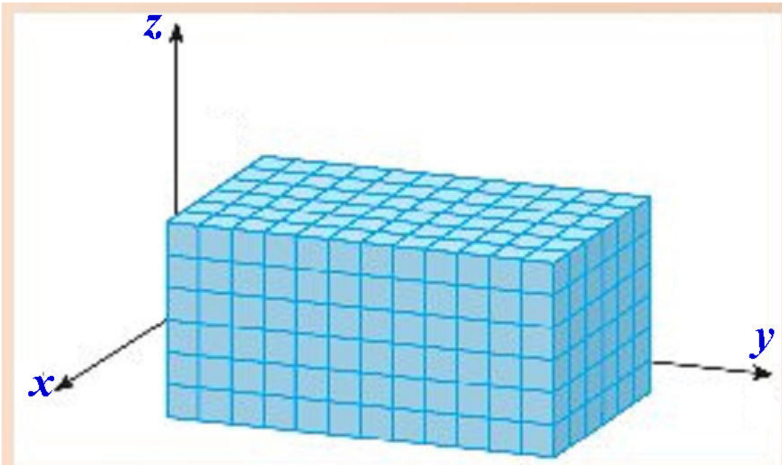


The exact value of the total mass is the limit of the sum as the number of sections goes to infinity.

$$\iint_R f(x, y)\, dx\, dy = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i,\ y_j)\, \Delta x\, \Delta y$$

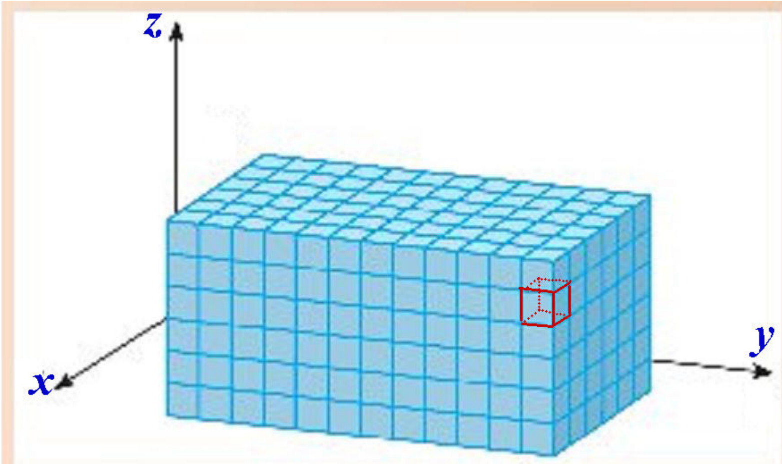
We may generalize to three dimensions.

Let $f(x, y, z)$ be the density at (x, y, z) in kg/m^3
Suppose we wish to find the total mass in a three dimensional region B.



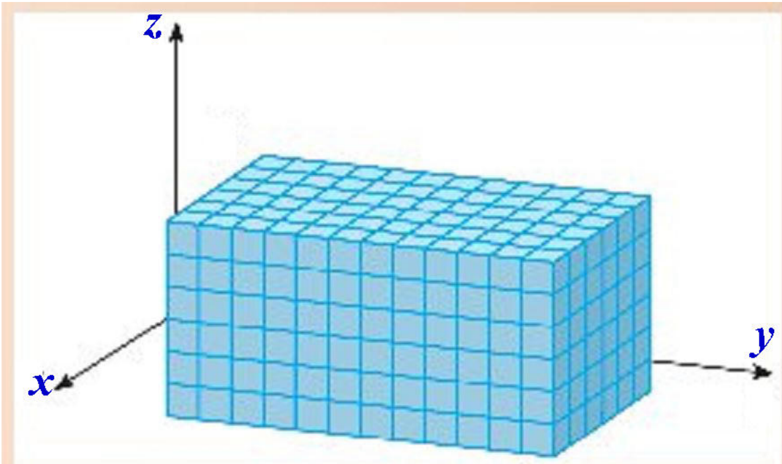
The mass of one section is (approximately)

$$f(x_i,\ y_j,\ z_k)\,\Delta x\,\Delta y\,\Delta z$$



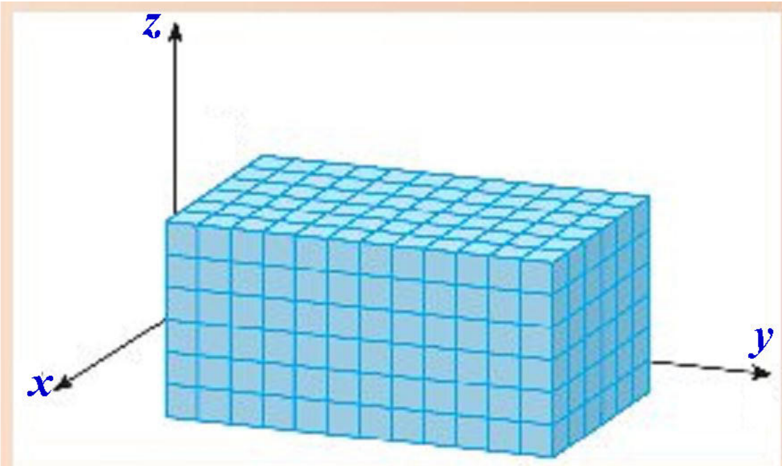
The total approximate mass is:

$$\sum_{i=1}^{\ell}\sum_{j=1}^{n}\sum_{k=1}^{m} f(x_i,\ y_j,\ z_k)\,\Delta x\,\Delta y\,\Delta z$$



The exact mass in region B is the limit of the sum as the number of sections goes to infinity.

$$\lim_{\ell,n,m\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{n} \sum_{k=1}^{m} f(x_i,\ y_j,\ z_k)\,\Delta x\,\Delta y\,\Delta z$$



The limit of the sum is a triple integral

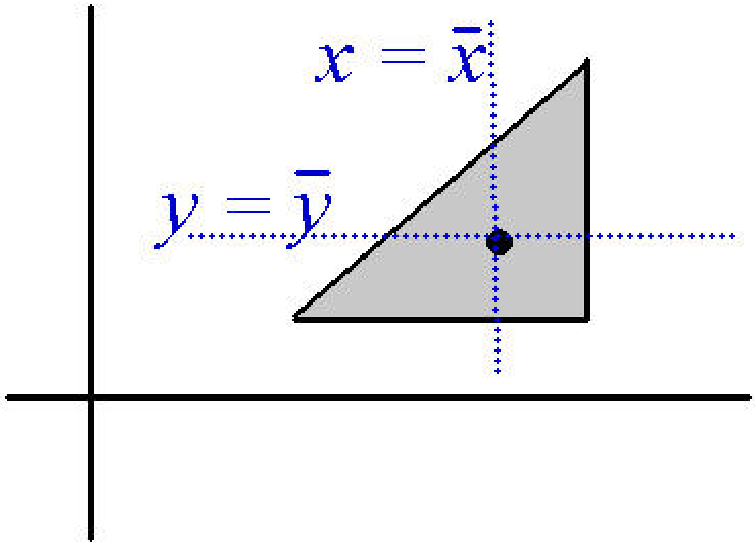
$$\text{Mass}(B) = \iiint_B f(x, y, z)\, dx\, dy\, dz$$

The limit of the sum is a triple integral

$$\text{Mass}(B) = \iiint_B f(x, y, z)\, dV$$

Centroid

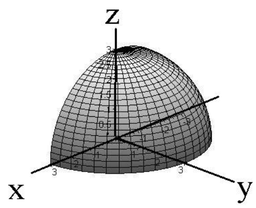
$$\overline{x} = \frac{1}{\text{Area}(R)} \iint_R x\, dA \qquad \overline{y} = \frac{1}{\text{Area}(R)} \iint_R y\, dA$$



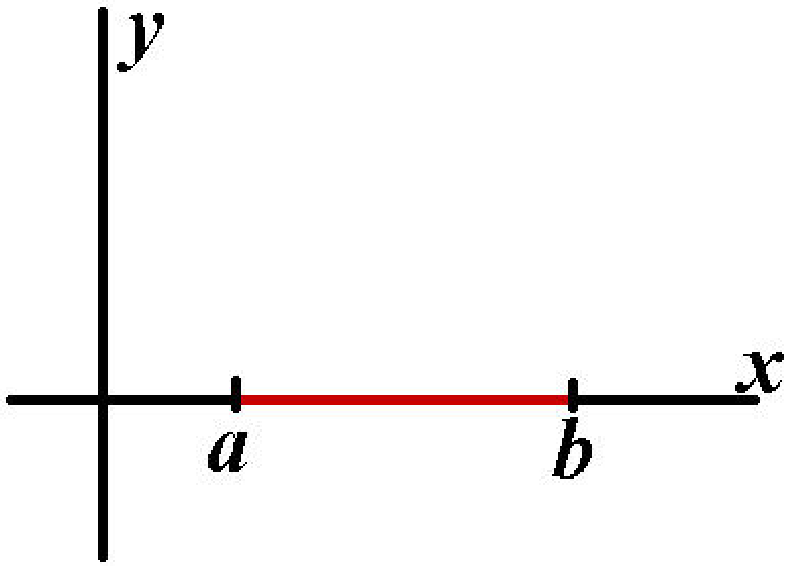
Centroid of a 3 dimensional region T

$$\overline{x} = \frac{1}{\textbf{Vol}(T)} \iiint_T x\,dV \quad \overline{y} = \frac{1}{\textbf{Vol}(T)} \iiint_T y\,dV$$

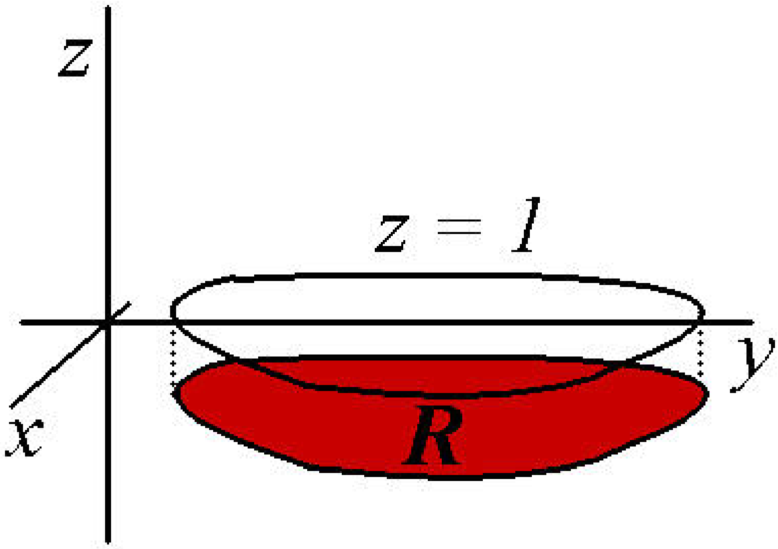
$$\overline{z} = \frac{1}{\textbf{Vol}(T)} \iiint_T z\,dV$$



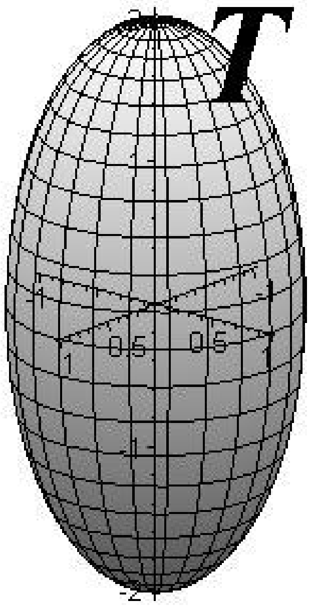
$$\int_a^b 1\,dx = b - a$$



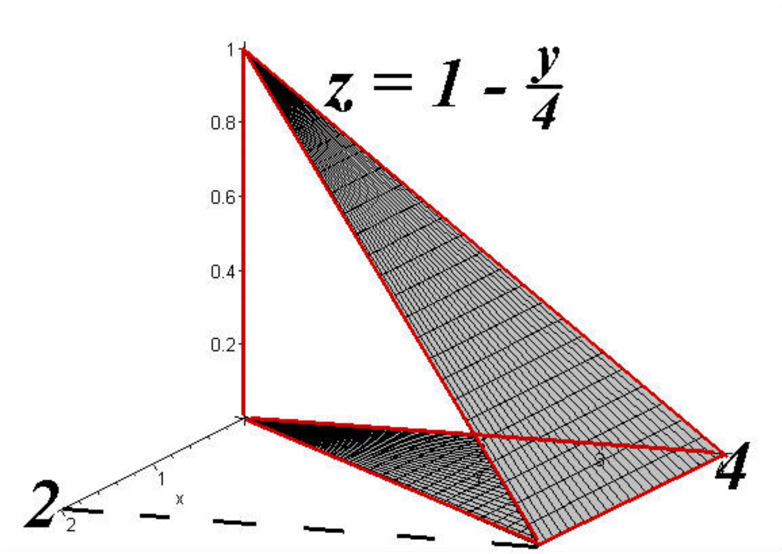
$$\mathbf{Area}(R) = \iint_R 1\, dA$$



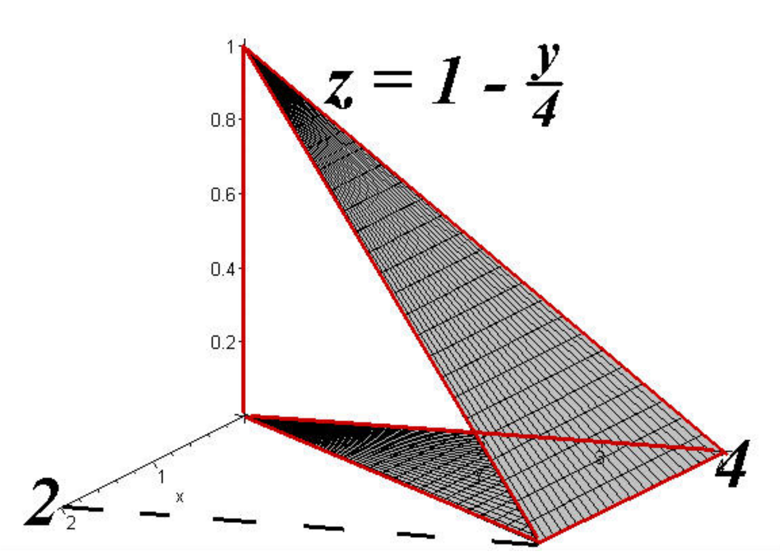
$$\mathbf{Vol}(T) = \iiint_T 1 \, dV$$



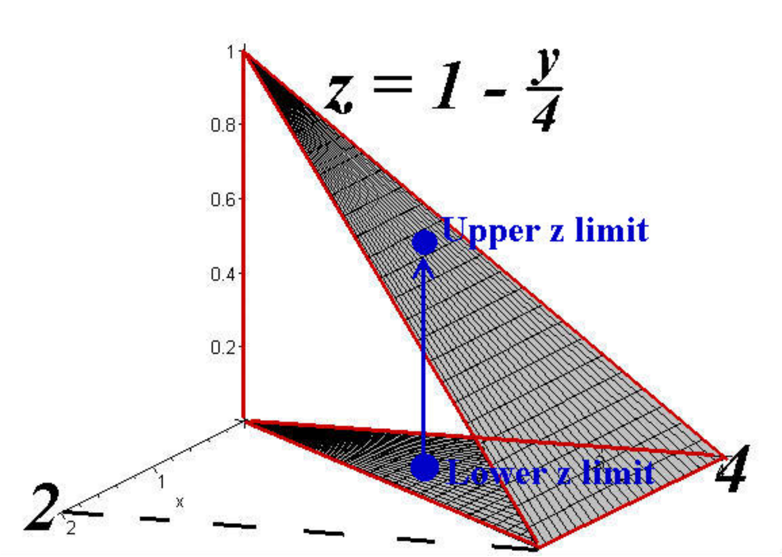
Find the volume under the portion of $z = 1 - \frac{y}{4}$ that is directly over the triangle with vertices $(0, 0, 0)$, $(2, 4, 0)$ and $(0, 4, 0)$



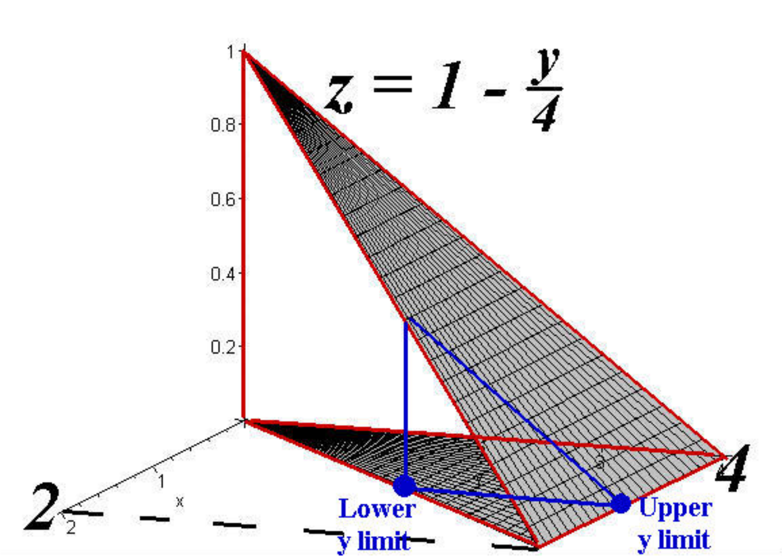
$$V = \int \int \int_T 1 \, dz \, dy \, dx$$



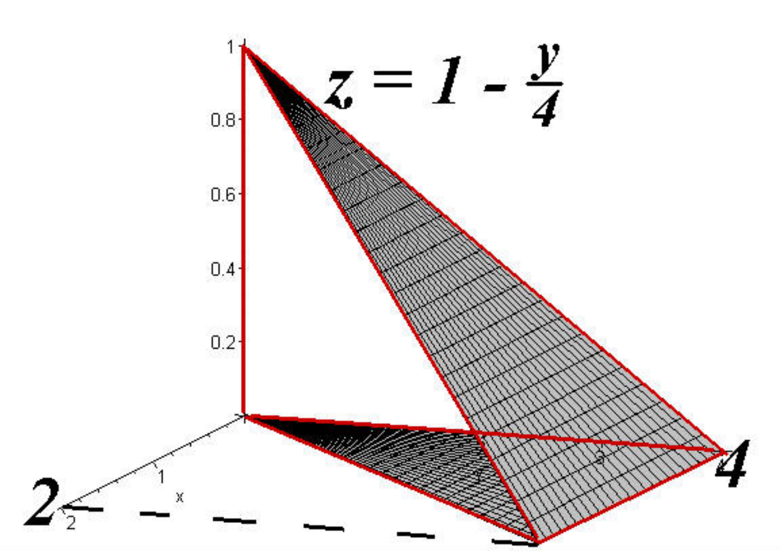
$$\int_{?}^{?}\int_{?}^{?}\int_{0}^{1-y/4} 1\,dz\,dy\,dx$$



$$V = \int_{?}^{?} \int_{2x}^{4} \int_{0}^{1-y/4} 1 \, dz \, dy \, dx$$



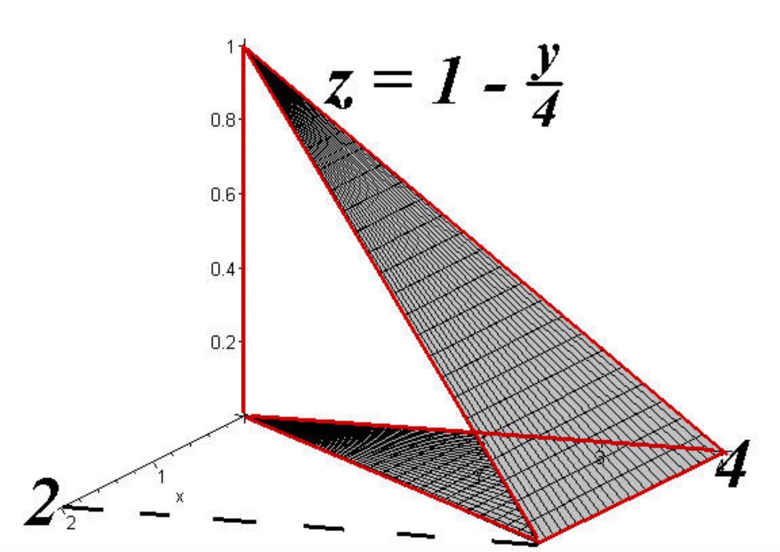
$$V = \int_0^2 \int_{2x}^4 \int_0^{1-y/4} 1 \, dz \, dy \, dx$$



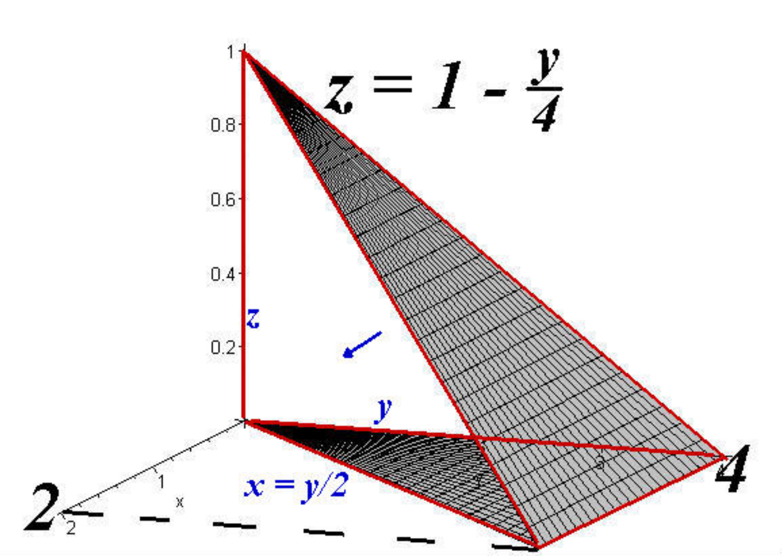
$$
\begin{aligned}
V &= \int_0^2 \int_{2x}^4 \int_0^{1-y/4} 1\, dz\, dy\, dx \\
&= \int_0^2 \int_{2x}^4 \left(1 - \frac{y}{4}\right)\, dy\, dx
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_0^2 \int_{2x}^4 \int_0^{1-y/4} 1 \, dz \, dy \, dx \\
&= \int_0^2 \int_{2x}^4 \left(1 - \frac{y}{4}\right) \, dy \, dx \\
&= \frac{4}{3}
\end{aligned}
$$

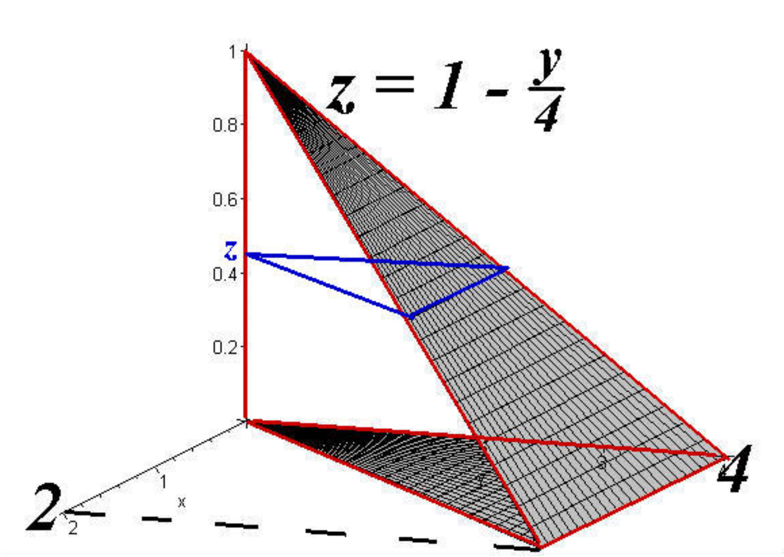
$$V = \int \int \int_T 1 \, dx \, dy \, dz$$



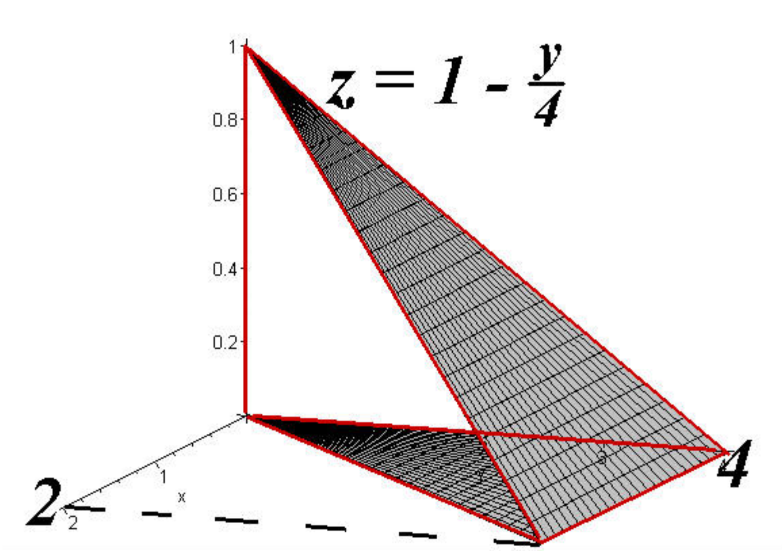
$$V = \int_{?}^{?} \int_{?}^{?} \int_{0}^{y/2} 1 \, dx \, dy \, dz$$



$$V = \int_{?}^{?} \int_{0}^{4-4z} \int_{0}^{y/2} 1 \, dx \, dy \, dz$$



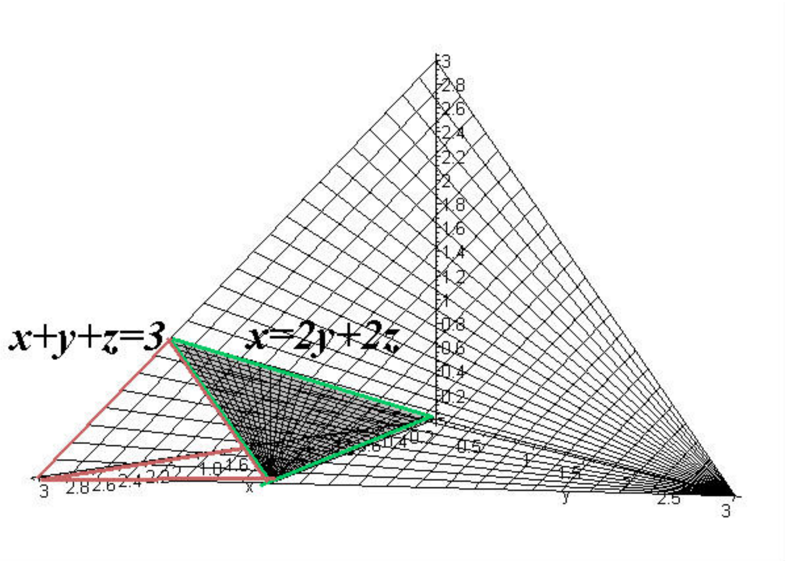
$$V = \int_0^1 \int_0^{4-4z} \int_0^{y/2} 1 \, dx \, dy \, dz$$



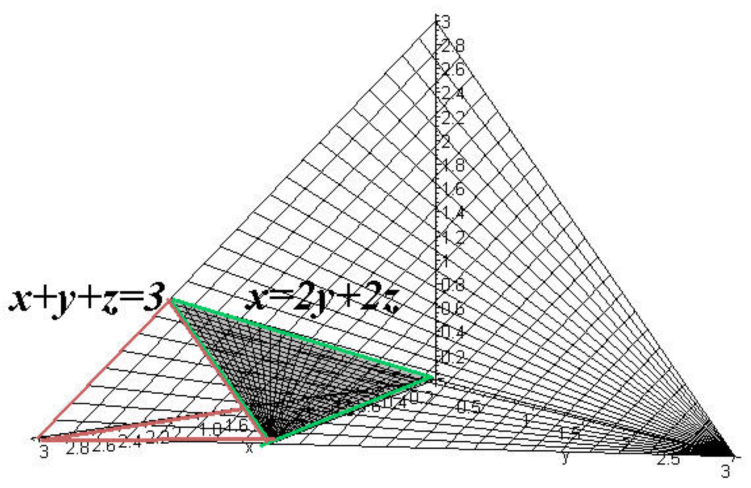
$$
\begin{aligned}
V &= \int_0^1 \int_0^{4-4z} \int_0^{y/2} 1\,dx\,dy\,dz \\
&= \int_0^1 \int_0^{4-4z} \frac{y}{2}\,dy\,dz \\
&= \int_0^1 \left[\frac{1}{4}y^2\right]_{y=0}^{4-4z} dz \\
&= \int_0^1 \frac{1}{4}(4-4z)^2\,dz \\
&= \frac{4}{3}
\end{aligned}
$$

Find the volume bounded by the planes:

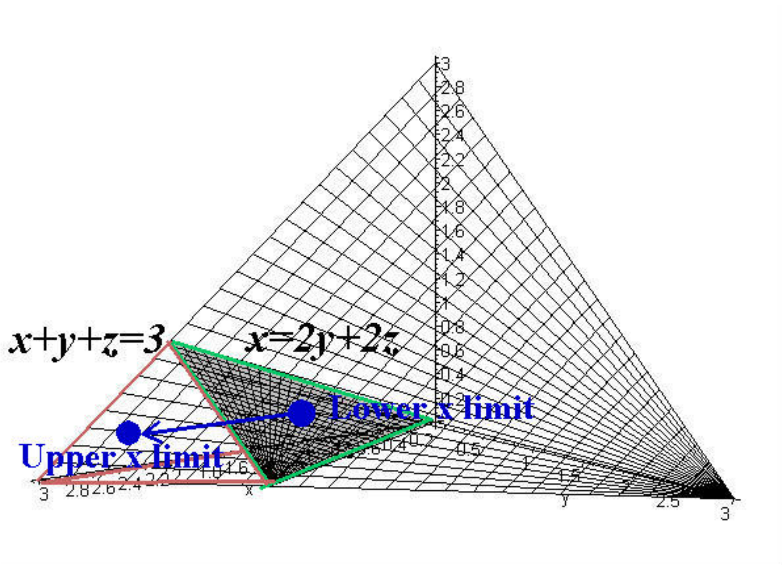
$$y = 0, \qquad z = 0 \qquad x = 2y + 2z \qquad x + y + z = 3$$



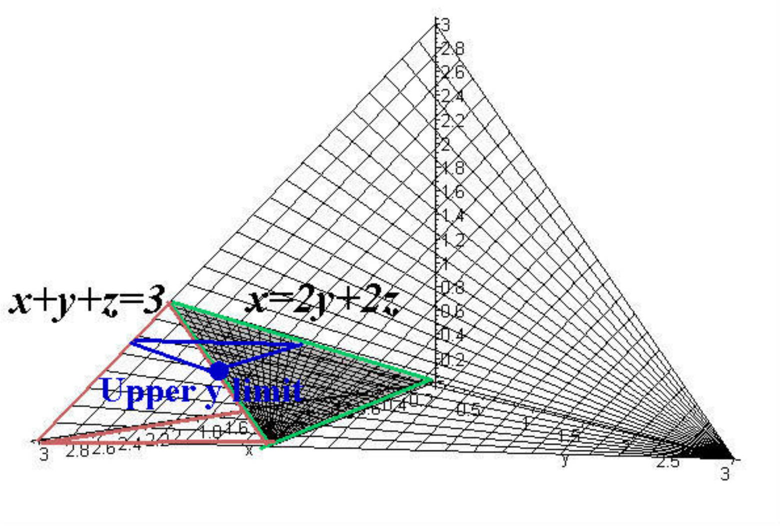
$$V = \iiint 1 \, dx \, dy \, dz$$



$$V = \int_{?}^{?} \int_{?}^{?} \int_{2y+2z}^{3-y-z} 1 \, dx \, dy \, dz$$



$$V=\int_{?}^{?}\int_{0}^{1-z}\int_{2y+2z}^{3-y-z}1\,dx\,dy\,dz$$



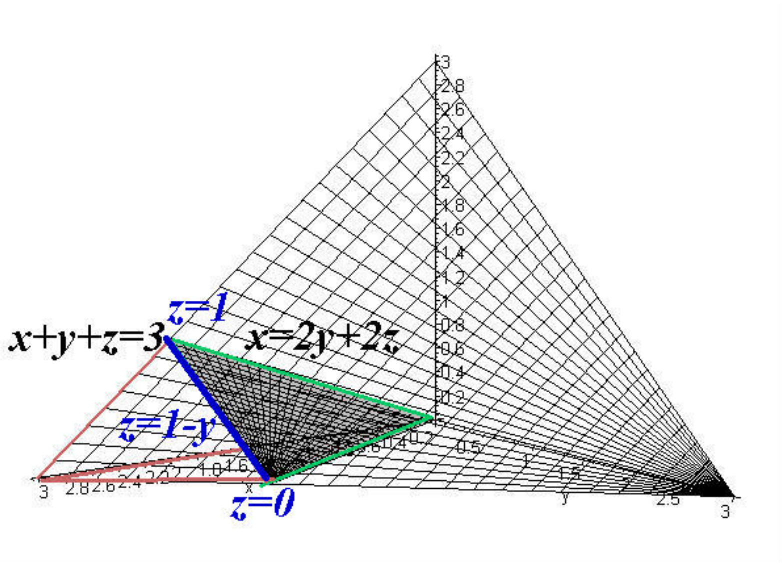
The plane $x = 2y + 2z$ intersects $x + y + z = 3$ when:

$$(2y + 2z) + y + z = 3$$

$$3y + 3z = 3$$

$$y = 1 - z$$

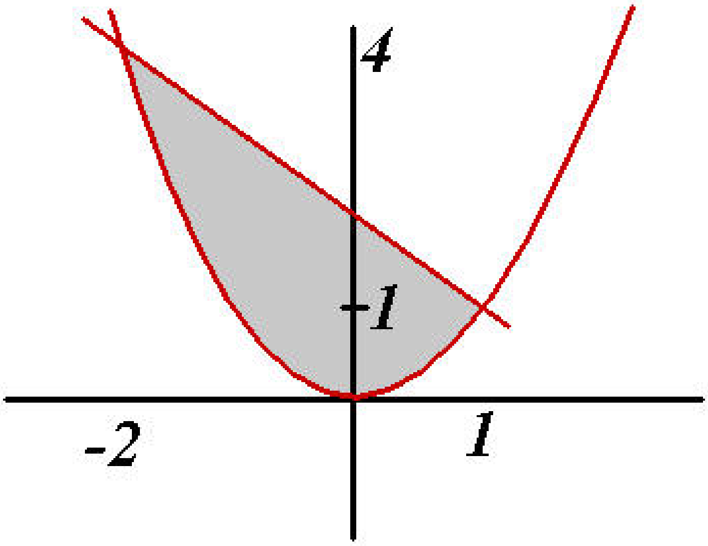
$$V = \int_0^1 \int_0^{1-z} \int_{2y+2z}^{3-y-z} 1\, dx\, dy\, dz$$



$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-z} \int_{2y+2z}^{3-y-z} 1 \, dx \, dy \, dz \\
&= \int_0^1 \int_0^{1-z} (3 - 3y - 3z) \, dy \, dz \\
&= \int_0^1 \left(\frac{3}{2} - 3z + \frac{3}{2} z^2 \right) \, dz \\
&= \frac{1}{2}
\end{aligned}
$$

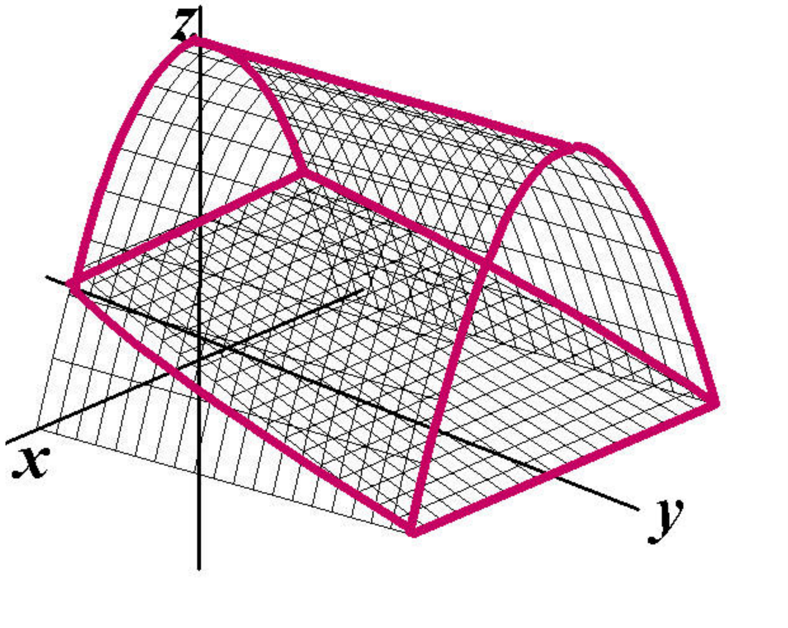
$$\iint_R f(x,y)\,dy\,dx$$

R is a two-dimensional region



$$\iiint_T f(x,y,z)\,dz\,dy\,dx$$

T is a three-dimensional region



For a double integral, there are two possible orders of integration

$$\int\int f(x, y)\, dx\, dy$$

$$\int\int f(x, y)\, dy\, dx$$

For a triple integral, there are six possible orders of integration

$$\iiint f(x,y,z)\,dx\,dy\,dz \qquad \iiint f(x,y,z)\,dx\,dz\,dy$$

$$\iiint f(x,y,z)\,dy\,dx\,dz \qquad \iiint f(x,y,z)\,dy\,dz\,dx$$

$$\iiint f(x,y,z)\,dz\,dx\,dy \qquad \iiint f(x,y,z)\,dz\,dy\,dx$$