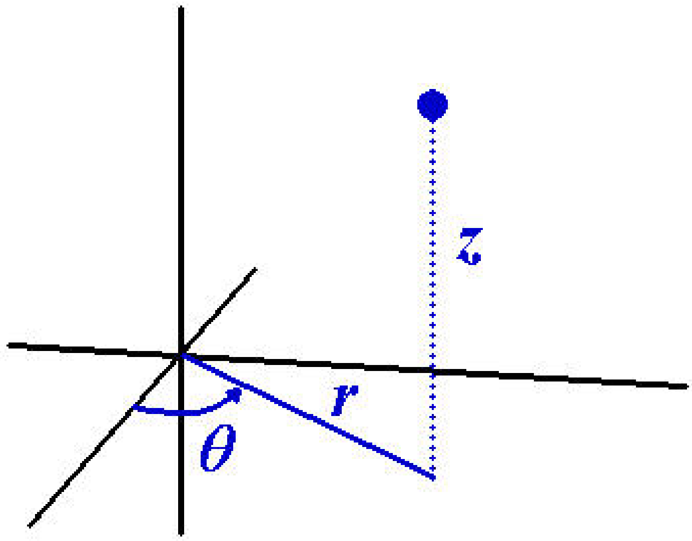
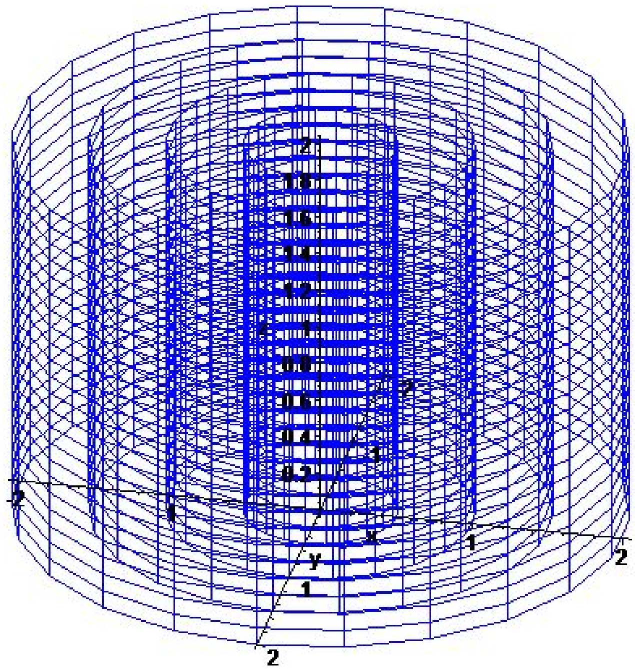
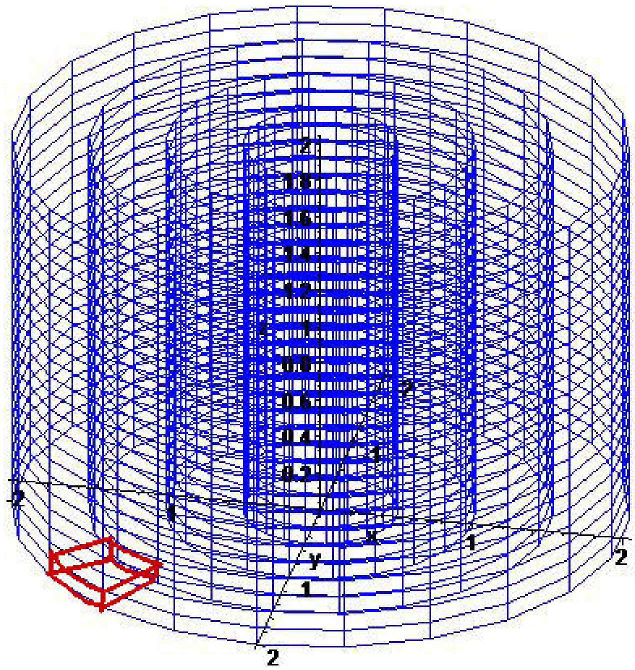
# Triple Integrals - Cylindrical Coordinates

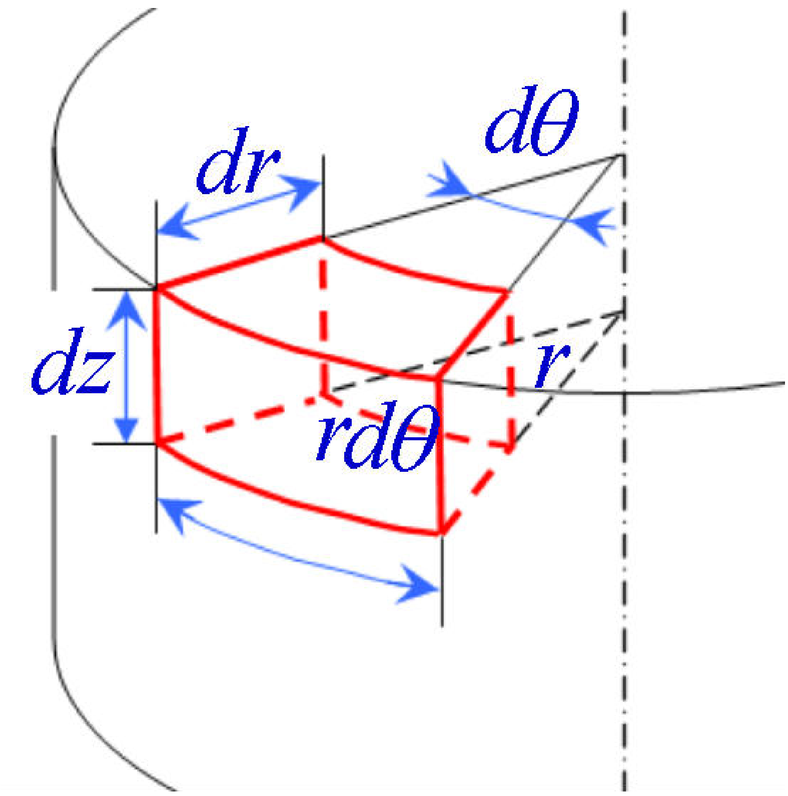




x
y
1
2

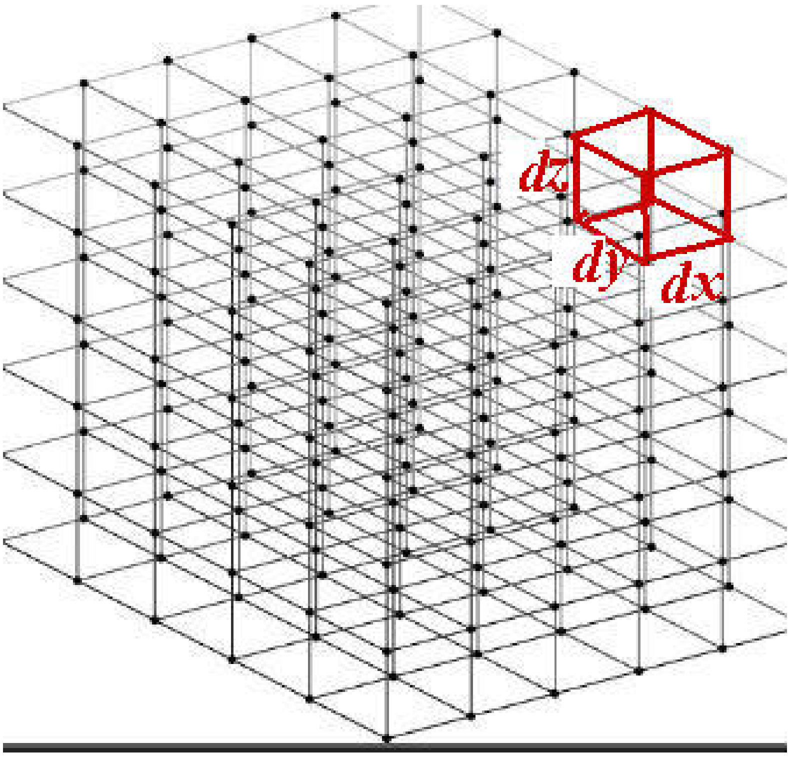


2
1
x
y
1
2
-2
2

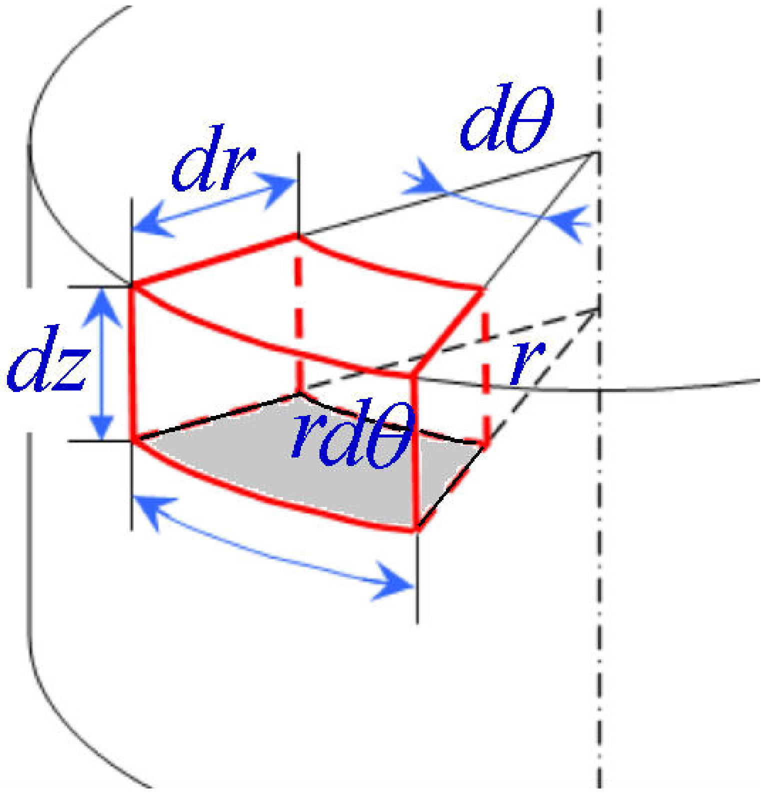


dr
dθ
dz
r
rdθ

$$dV = dx\, dy\, dz$$

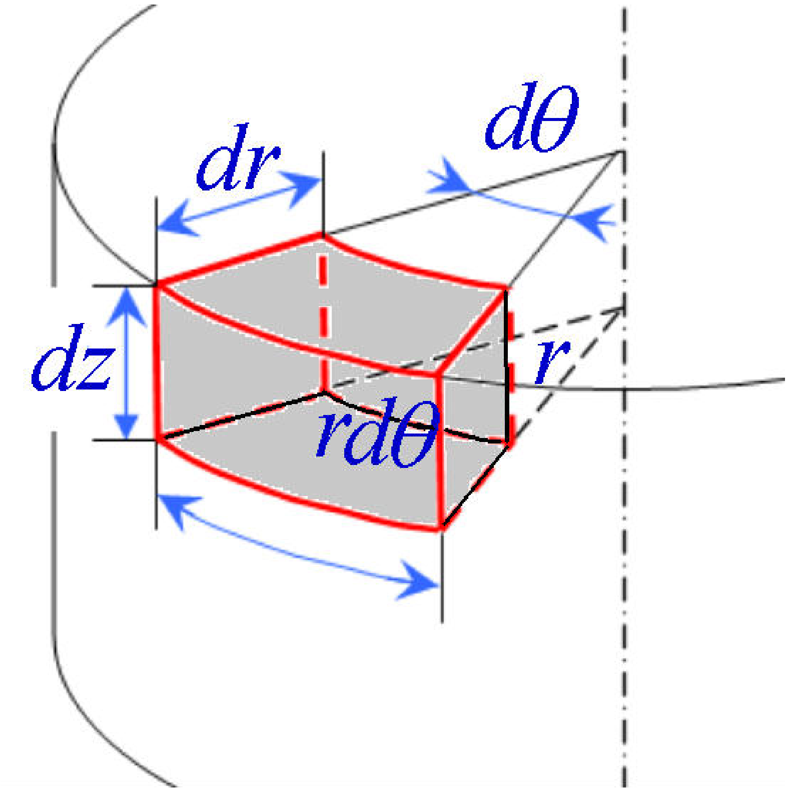


The area at the base is $dA = r\,d\theta\,dr$



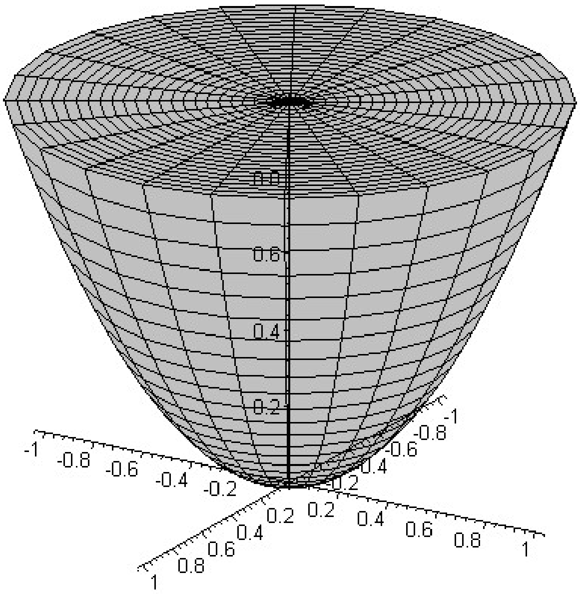
The volume of a cylindrical section:

$$dV = dA \cdot dz = r\,d\theta\,dr\,dz$$

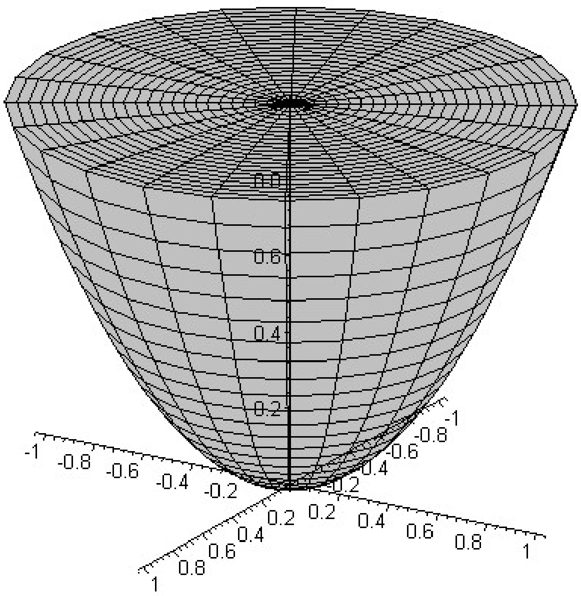


**Example:**

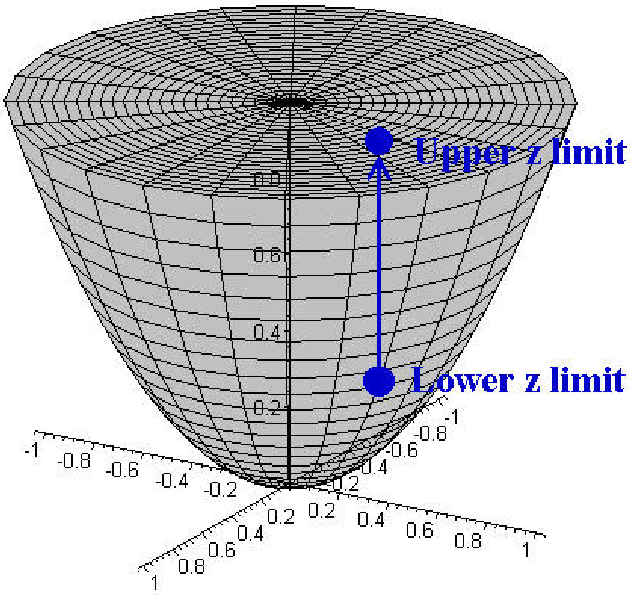
Find the volume of the region bounded from below by $z = x^2 + y^2$ and from above by $z = 1$.



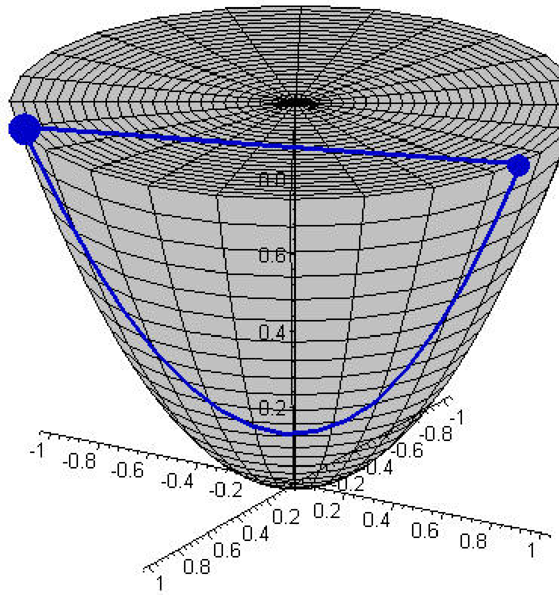
$$V = \iiint 1 \, dz \, dy \, dx$$



$$V = \int_{?}^{?}\int_{?}^{?}\int_{x^2+y^2}^{1} 1\, dz\, dy\, dx$$



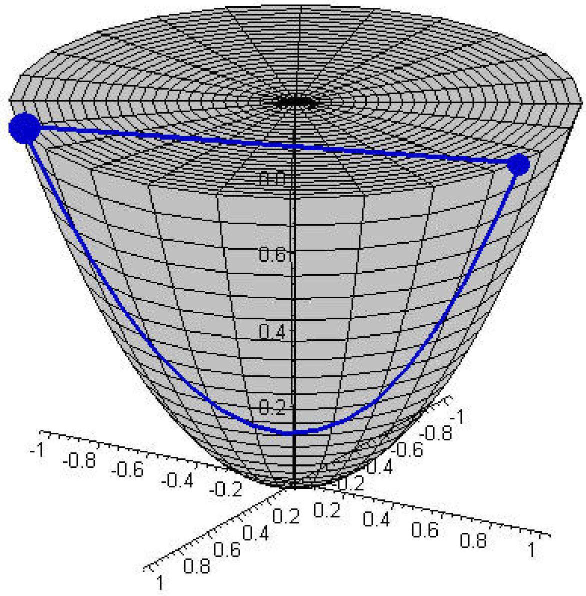
$$V = \int_{?}^{?}\int_{?}^{?}\int_{x^2+y^2}^{1} 1\,dz\,dy\,dx$$



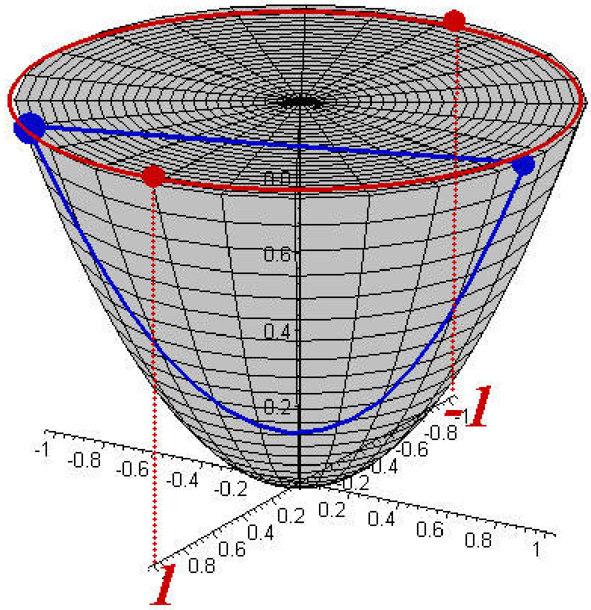
The plane $z = 1$ intersects the paraboloid $z = x^2 + y^2$ when $x^2 + y^2 = 1$ so:

$$y = \pm\sqrt{1 - x^2}$$

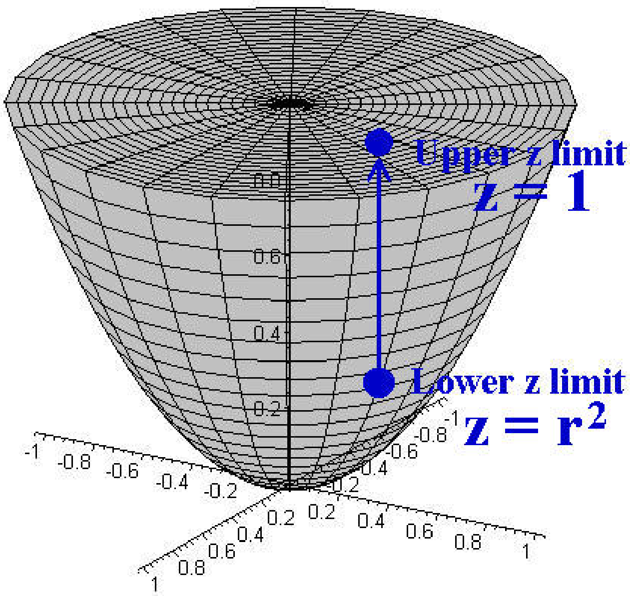
$$V = \int_{?}^{?} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{1} 1 \, dz \, dy \, dx$$



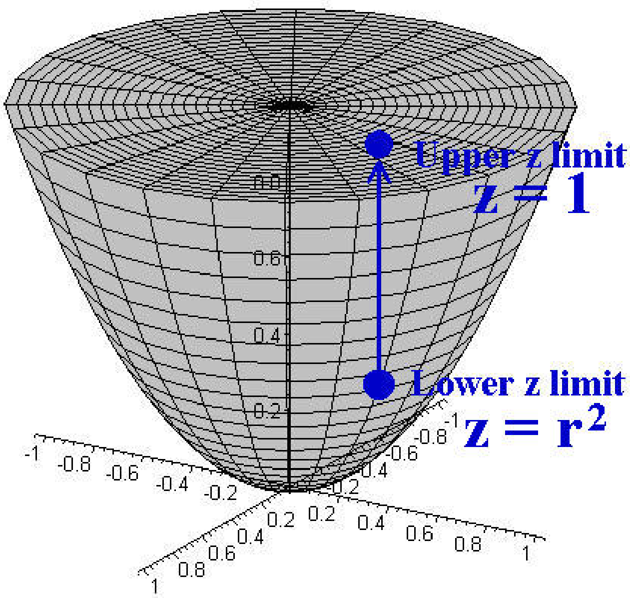
$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{1} 1\, dz\, dy\, dx$$



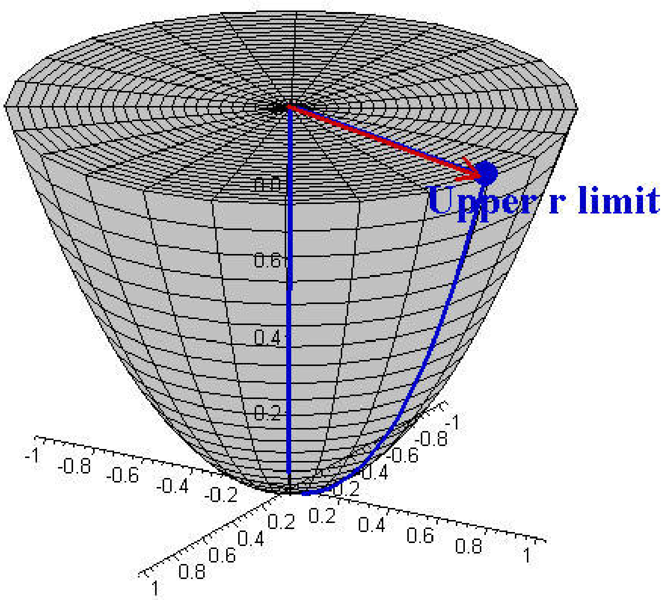
$$V = \iiint 1\,dV = \iiint 1\,r\,dz\,dr\,d\theta$$



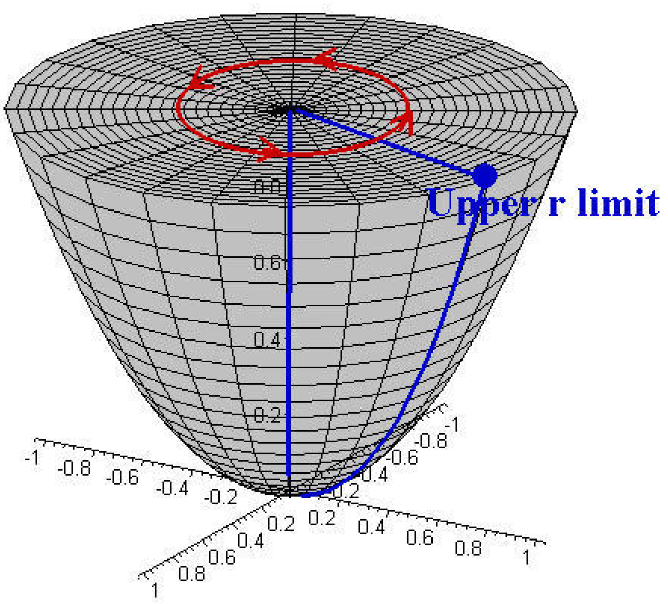
$$V = \iiint 1\, dV = \iiint_{r^2}^{1} 1\, r\, dz\, dr\, d\theta$$



$$V = \int_{?}^{?} \int_{0}^{1} \int_{r^2}^{1} r \, dz \, dr \, d\theta$$



$$V = \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\,dz\,dr\,d\theta$$

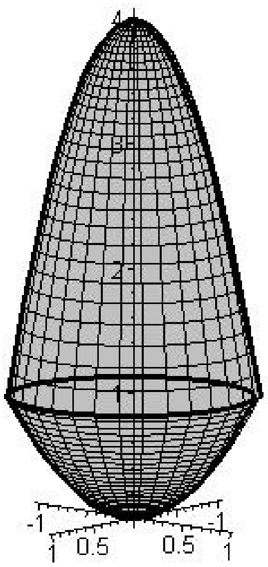


$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \Big[ r z \Big]_{z=r^2}^{1} \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left( r - r^3 \right) \, dr \, d\theta
\end{aligned}
$$

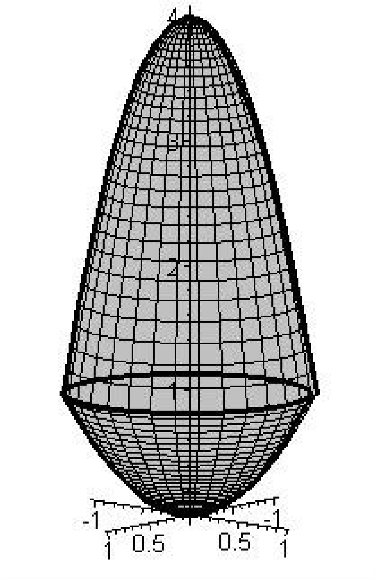
$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \Big[ r z \Big]_{z=r^2}^1 \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left( r - r^3 \right) \, dr \, d\theta \\
&= \int_0^{2\pi} \left[ \frac{1}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^1 d\theta
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \Big[ rz \Big]_{z=r^2}^1 \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left( r - r^3 \right) \, dr \, d\theta \\
&= \int_0^{2\pi} \left[ \frac{1}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^1 d\theta \\
&= \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{\pi}{2}
\end{aligned}
$$

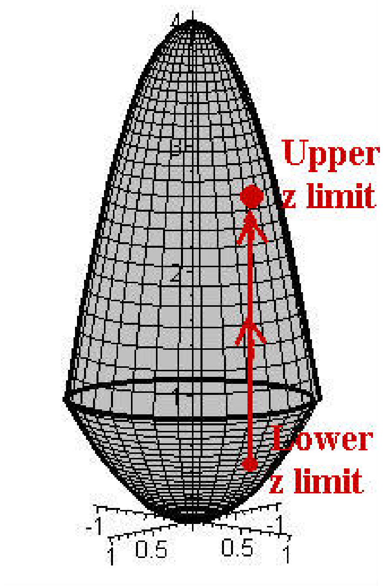
Let $T$ be the region bounded by $z = x^2 + y^2$ and $z = 4 - 3(x^2 + y^2)$. Calculate the triple integral $\iiint_T 1 \, dV$



$$V = \iiint_T 1\, r\, dz\, dr\, d\theta$$



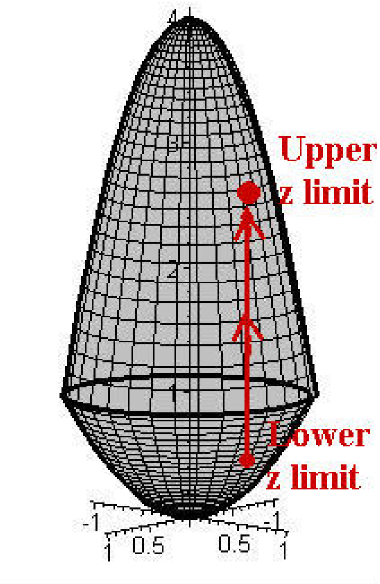
$$V = \iiint_T 1\, r\, dz\, dr\, d\theta$$



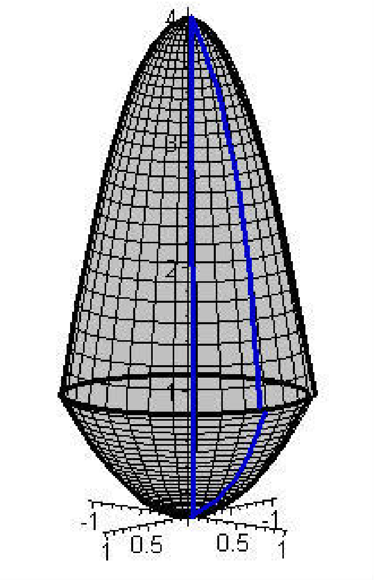
The lower paraboloid is $z = x^2 + y^2 = r^2$

The upper paraboloid is $z = 4 - 3\left(x^2 + y^2\right) = 4 - 3r^2$

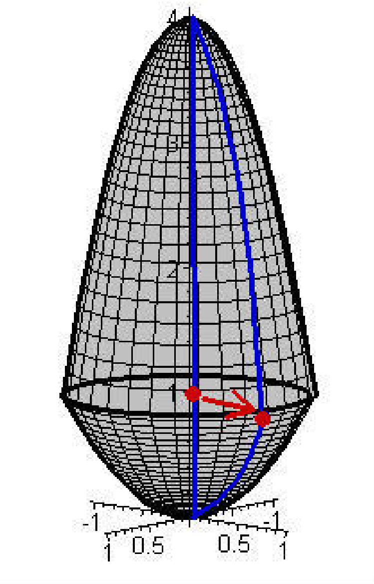
$$V = \iiint_{r^2}^{4-3r^2} 1\, r\, dz\, dr\, d\theta$$



$$V = \int\int\int_{r^2}^{4-3r^2} 1\, r\, dz\, dr\, d\theta$$



$$V = \iiint_{r^2}^{4-3r^2} 1\, r\, dz\, dr\, d\theta$$



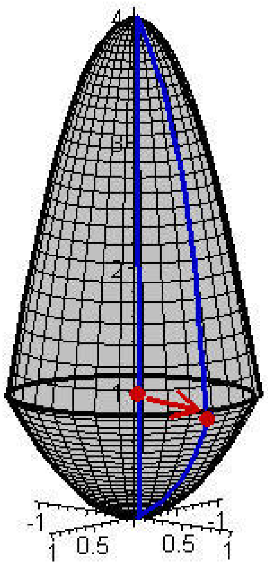
The paraboloid $z = r^2$ intersects the paraboloid $z = 4 - 3r^2$ when:

$$r^2 = 4 - 3r^2$$

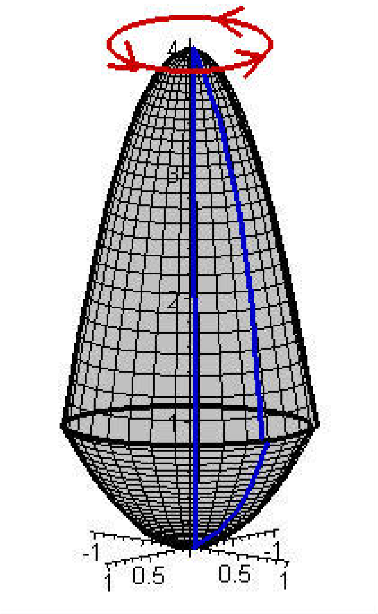
$$4r^2 = 4$$

$$r = 1$$

$$V = \iint_0^1 \int_{r^2}^{4-3r^2} 1\, r\, dz\, dr\, d\theta$$



$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{4-3r^2} 1\, r\, dz\, dr\, d\theta$$



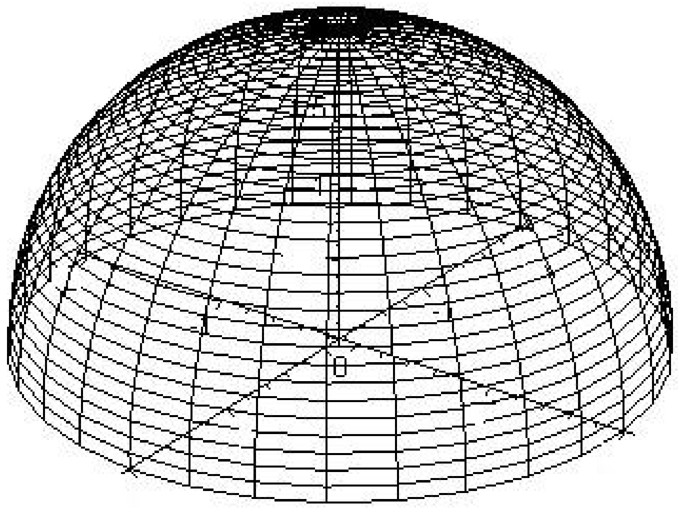
$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^1\int_{r^2}^{4-3r^2} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \Big[rz\Big]_{z=r^2}^{4-3r^2} dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(4r-4r^3\right) dr\,d\theta
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^1\int_{r^2}^{4-3r^2} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \Big[rz\Big]_{z=r^2}^{4-3r^2} dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(4r-4r^3\right) dr\,d\theta \\
&= 2\pi
\end{aligned}
$$

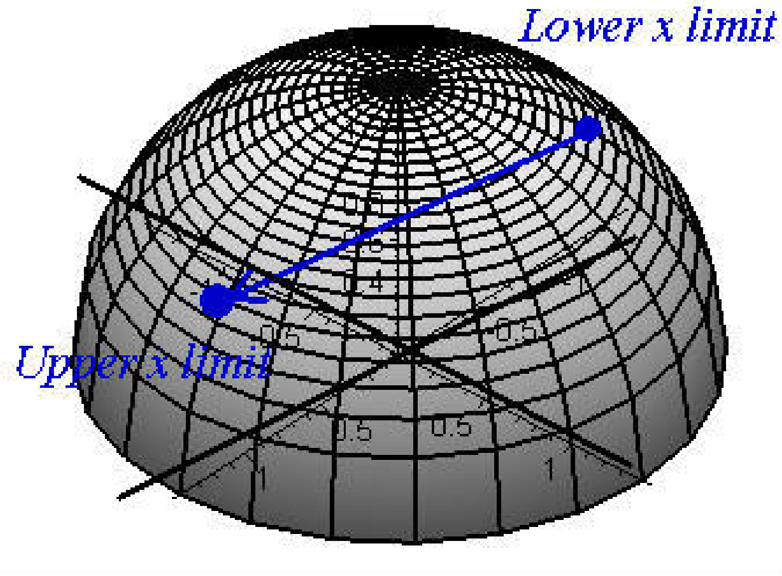
Let $\mathcal{H}$ be the hemispherical region above the $xy$ plane but inside the sphere $x^2 + y^2 + z^2 = 1$.
Express the volume of $\mathcal{H}$ as $\iiint_{\mathcal{H}} 1\, dV$

a) In rectangular coordinates

b) In cylindrical coordinates



$$\text{Vol}(\mathcal{H}) = \int_?^? \int_?^? \int_?^? 1 \, dx \, dy \, dz$$

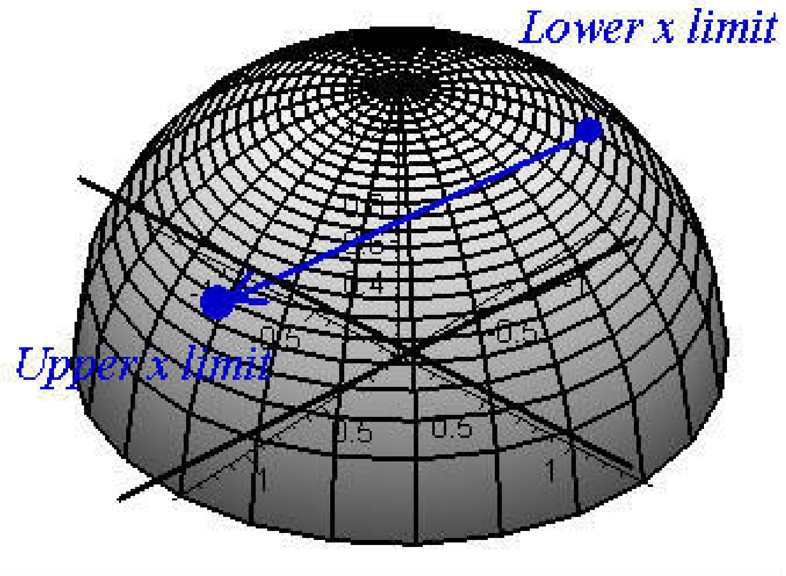


$$x^2 + y^2 + z^2 = 1$$

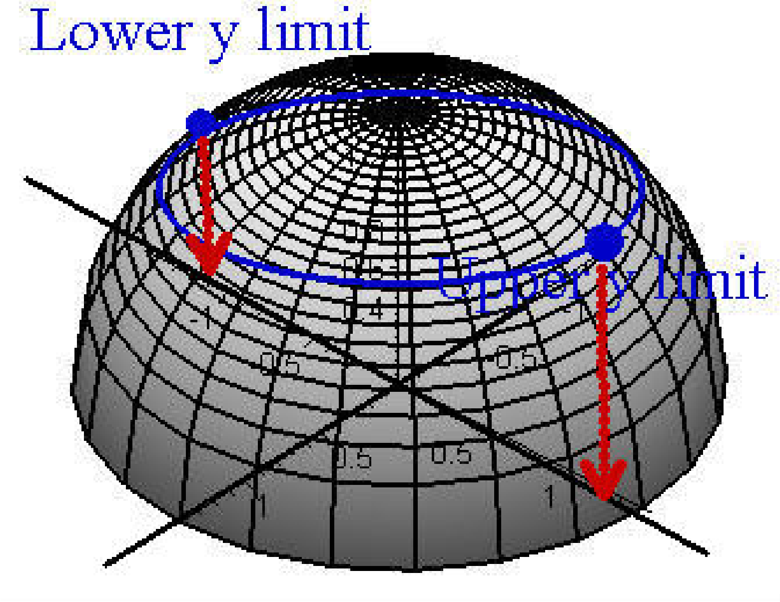
$$x^2 = 1 - y^2 - z^2$$

$$x = \pm\sqrt{1 - y^2 - z^2}$$

$$\mathrm{Vol}(\mathcal{H}) = \int_{?}^{?} \int_{?}^{?} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 1 \, dx \, dy \, dz$$



$$\mathrm{Vol}(\mathcal{H}) = \int_?^? \int_?^? \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 1 \, dx \, dy \, dz$$

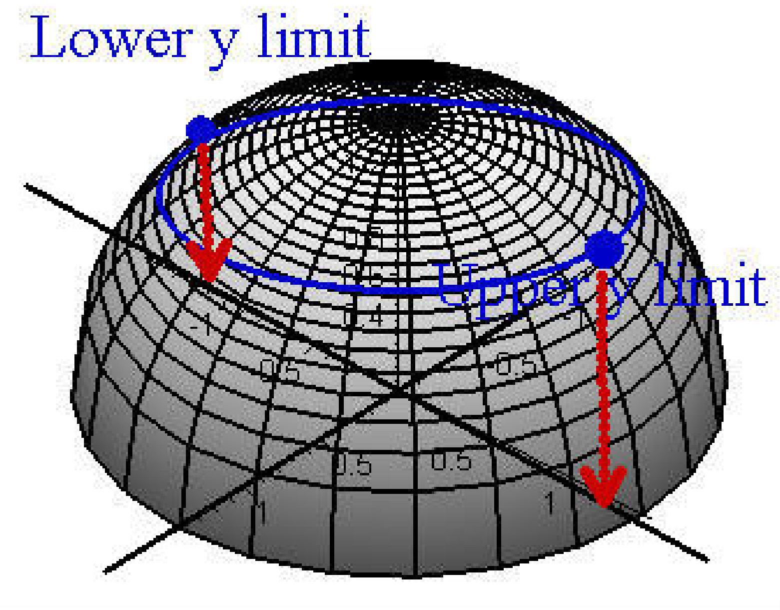


In the $yz$-plane, the $x$ value is 0 so $x^2 + y^2 + z^2 = 1$ becomes:

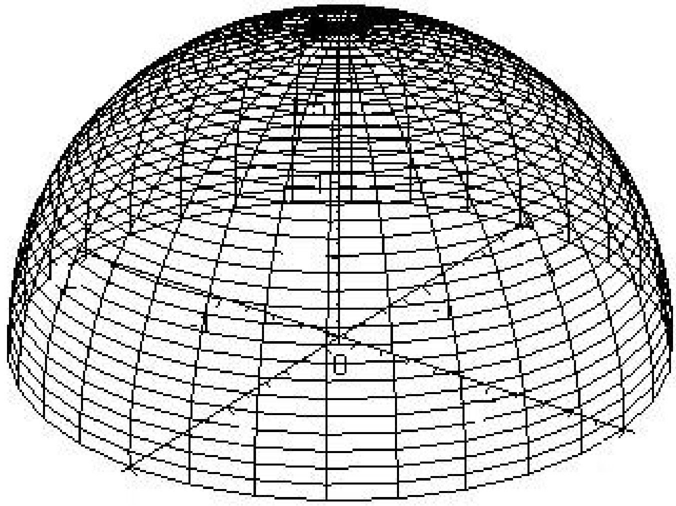
$$y^2 + z^2 = 1$$

$$y = \pm\sqrt{1 - z^2}$$

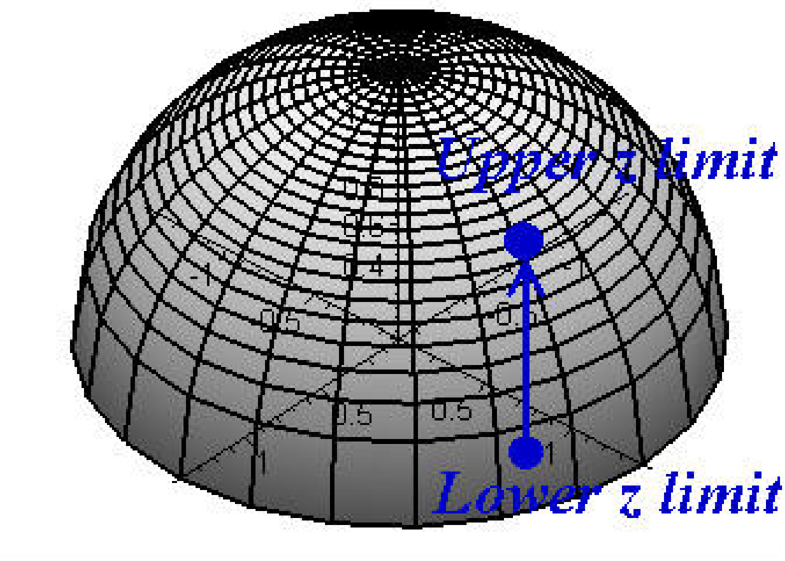
$$\mathrm{Vol}(\mathcal{H}) = \int_{?}^{?} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 1 \, dx \, dy \, dz$$



$$\mathrm{Vol}(\mathcal{H}) = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 1 \, dx \, dy \, dz$$



$$\text{Vol}(\mathcal{H}) = \iiint_{\mathcal{H}} 1\, r\, dz\, d\theta\, dr$$

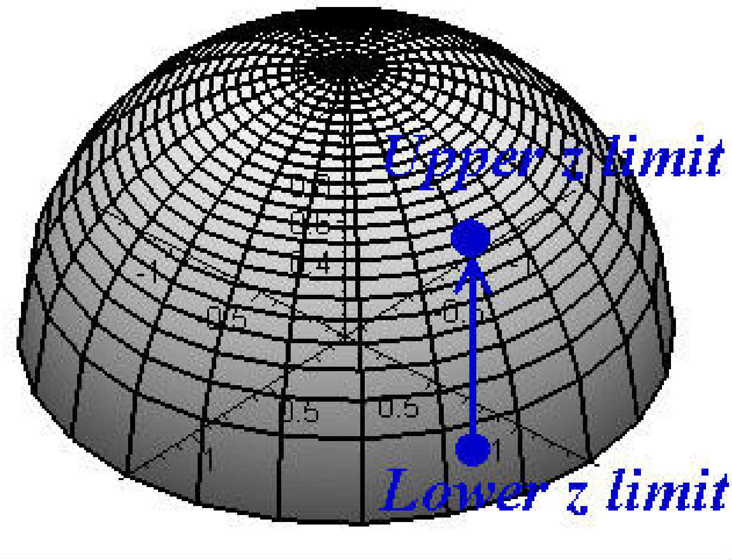


$$x^2 + y^2 + z^2 = 1$$

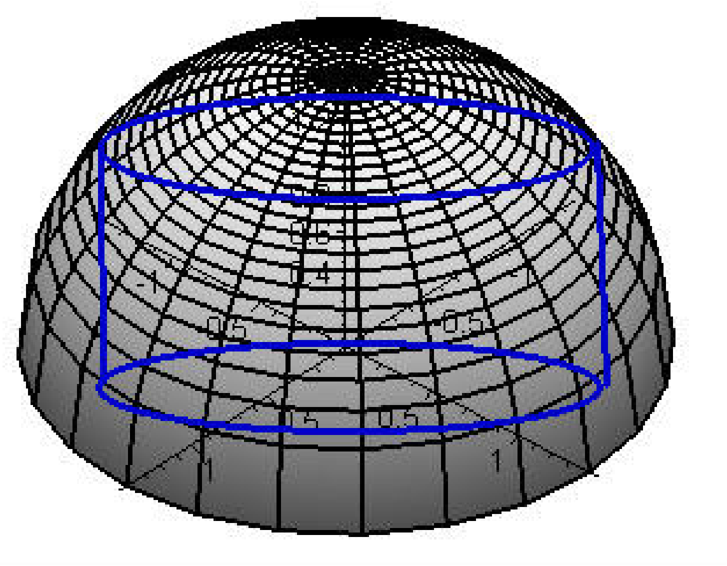
$$r^2 + z^2 = 1$$

$$z = \sqrt{1 - r^2}$$

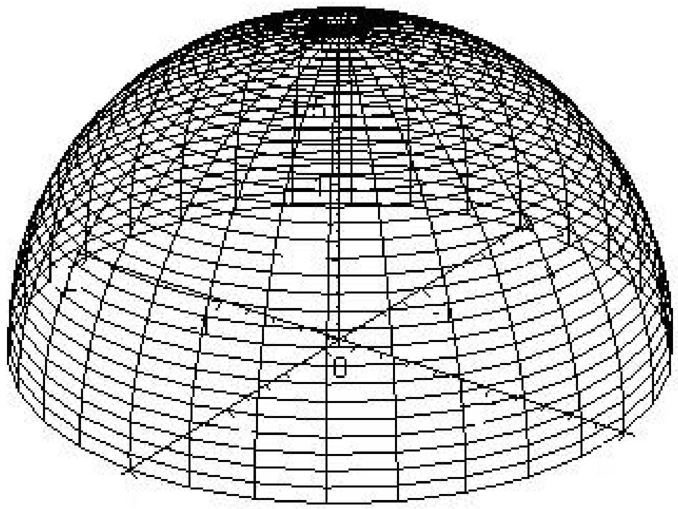
$$\mathrm{Vol}(\mathcal{H}) = \iiint_0^{\sqrt{1-r^2}} r\,dz\,d\theta\,dr$$



$$\mathrm{Vol}(\mathcal{H}) = \iint_0^{2\pi} \int_0^{\sqrt{1-r^2}} r \, dz \, d\theta \, dr$$



$$\mathrm{Vol}(\mathcal{H}) = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} r\, dz\, d\theta\, dr$$

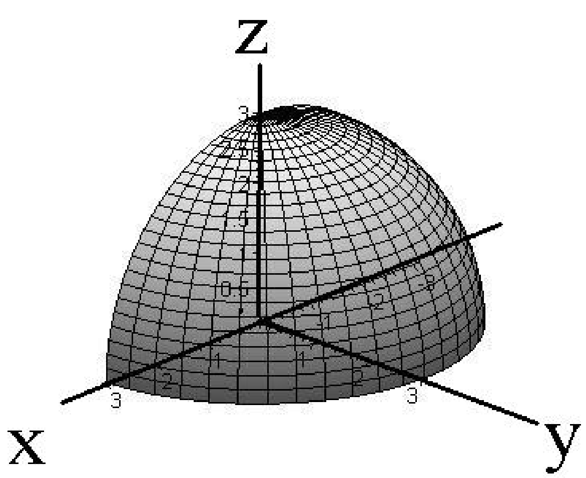


$$
\begin{aligned}
\mathrm{Vol}(\mathcal{H}) &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} r \, dz \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} r\sqrt{1-r^2} \, d\theta \, dr
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Vol}(\mathcal{H}) &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} r \, dz \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} r\sqrt{1-r^2} \, d\theta \, dr \\
&= \int_0^1 2\pi r\sqrt{1-r^2} \, dr
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Vol}(\mathcal{H}) &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} r\, dz\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} r\sqrt{1-r^2}\, d\theta\, dr \\
&= \int_0^1 2\pi r\sqrt{1-r^2}\, dr \\
&= \frac{2\pi}{3}
\end{aligned}
$$

Let $Q$ be the quarter sphere of radius 1 where $y,\ z \geq 0$. Find the coordinates of the centroid.



$$\overline{x} = \frac{1}{\text{Vol}(Q)} \iiint_Q x \, dV$$

$$\overline{y} = \frac{1}{\text{Vol}(Q)} \iiint_Q y \, dV$$

$$\overline{z} = \frac{1}{\text{Vol}(Q)} \iiint_Q z \, dV$$

The volume of $Q$ is $\frac{\pi}{3}$

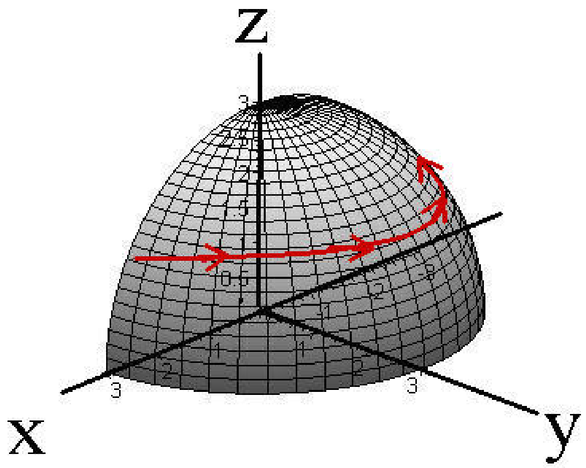
$$\overline{x} = \frac{3}{\pi} \iiint_Q x\,dV$$

$$\overline{y} = \frac{3}{\pi} \iiint_Q y\,dV$$

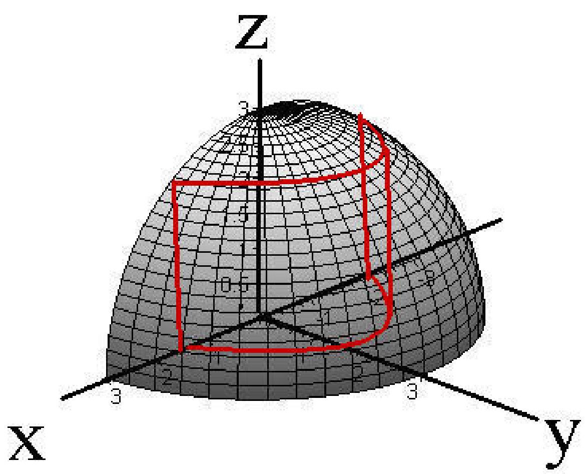
$$\overline{z} = \frac{3}{\pi} \iiint_Q z\,dV$$

$$\begin{aligned}\overline{y} &= \frac{3}{\pi}\iiint_Q y\,dV \\ &= \frac{3}{\pi}\iiint_Q (r\sin\theta)\cdot r\,d\theta\,dz\,dr \\ &= \frac{3}{\pi}\iiint_Q r^2\sin\theta\,d\theta\,dz\,dr\end{aligned}$$

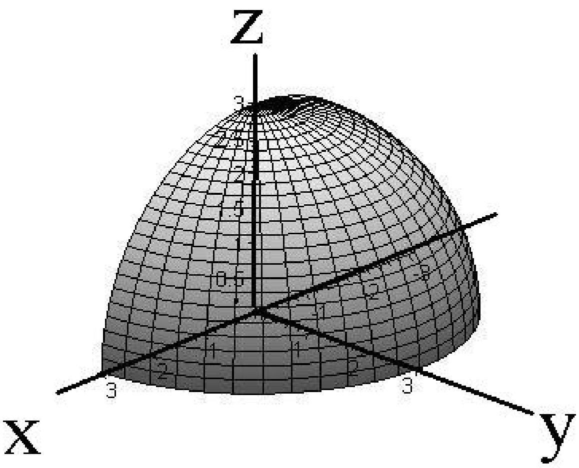
$$\overline{y} = \frac{3}{\pi} \int_{?}^{?} \int_{?}^{?} \int_{0}^{\pi} r^2 \sin\theta \, d\theta \, dz \, dr$$



$$\overline{y} = \frac{3}{\pi} \int_{?}^{?} \int_{0}^{\sqrt{1-r^2}} \int_{0}^{\pi} r^2 \sin\theta \, d\theta \, dz \, dr$$



$$\overline{y} = \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi} r^2 \sin\theta \, d\theta \, dz \, dr$$



$$\begin{aligned}\overline{y} &= \frac{3}{\pi}\int_0^1\int_0^{\sqrt{1-r^2}}\int_0^{\pi} r^2\sin\theta\,d\theta\,dz\,dr\\ &= \frac{3}{\pi}\int_0^1\int_0^{\sqrt{1-r^2}} 2r^2\,dz\,dr\end{aligned}$$

$$
\begin{aligned}
\overline{y} &= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi} r^2 \sin\theta \, d\theta \, dz \, dr \\
&= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} 2r^2 \, dz \, dr \\
&= \frac{6}{\pi} \int_0^1 r^2 \sqrt{1-r^2} \, dr
\end{aligned}
$$

$$
\begin{aligned}
\overline{y} &= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi} r^2 \sin\theta \, d\theta \, dz \, dr \\
&= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} 2r^2 \, dz \, dr \\
&= \frac{6}{\pi} \int_0^1 r^2 \sqrt{1-r^2} \, dr \\
&= \frac{3}{8}
\end{aligned}
$$

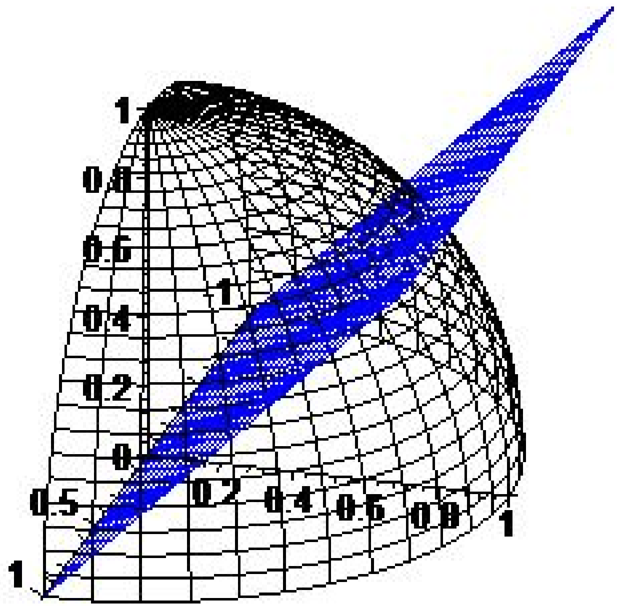
$$\overline{z} = \frac{3}{\pi} \iiint_Q z\,dV = \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi} zr\,d\theta\,dz\,dr$$

$$= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \pi zr\,dz\,dr$$

$$
\begin{aligned}
\overline{z} &= \frac{3}{\pi}\iiint_Q z\,dV = \frac{3}{\pi}\int_0^1\int_0^{\sqrt{1-r^2}}\int_0^{\pi} zr\,d\theta\,dz\,dr \\
&= \frac{3}{\pi}\int_0^1\int_0^{\sqrt{1-r^2}} \pi zr\,dz\,dr \\
&= 3\int_0^1 \left[\frac{1}{2}z^2 r\right]_{z=0}^{\sqrt{1-r^2}} dr \\
&= 3\int_0^1 \frac{1}{2}(r-r^3)\,dr
\end{aligned}
$$

$$
\begin{aligned}
\overline{z} &= \frac{3}{\pi} \iiint_Q z \, dV = \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi} zr \, d\theta \, dz \, dr \\
&= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \pi zr \, dz \, dr \\
&= 3 \int_0^1 \left[ \frac{1}{2} z^2 r \right]_{z=0}^{\sqrt{1-r^2}} dr \\
&= 3 \int_0^1 \frac{1}{2}(r - r^3) \, dr = \frac{3}{8}
\end{aligned}
$$

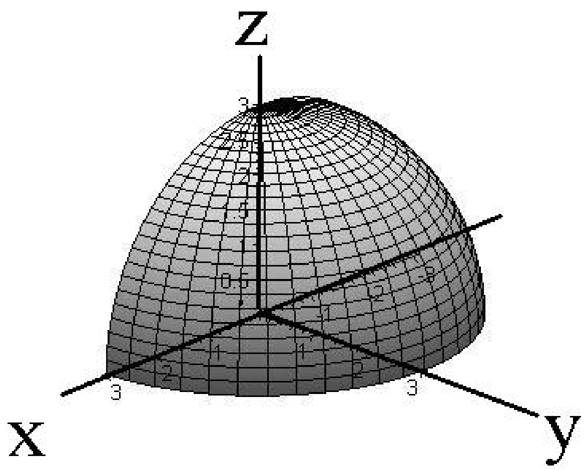
$$z = y$$



$$
\begin{aligned}
\overline{x} &= \frac{3}{\pi} \iiint_Q x \, dV \\
&= \frac{3}{\pi} \iiint_Q (r \cos\theta) \cdot r \, d\theta \, dz \, dr \\
&= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi} r^2 \cos\theta \, d\theta \, dz \, dr
\end{aligned}
$$

$$
\begin{aligned}
\overline{x} &= \frac{3}{\pi} \iiint_Q x \, dV \\
&= \frac{3}{\pi} \iiint_Q (r \cos\theta) \cdot r \, d\theta \, dz \, dr \\
&= \frac{3}{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi} r^2 \cos\theta \, d\theta \, dz \, dr \\
&= 0
\end{aligned}
$$

$$(\overline{x},\ \overline{y},\ \overline{z}) = \left(0,\ \frac{3}{8},\ \frac{3}{8}\right)$$



$$(\overline{x},\ \overline{y},\ \overline{z}) = \left(0,\ \frac{3}{8},\ \frac{3}{8}\right)$$