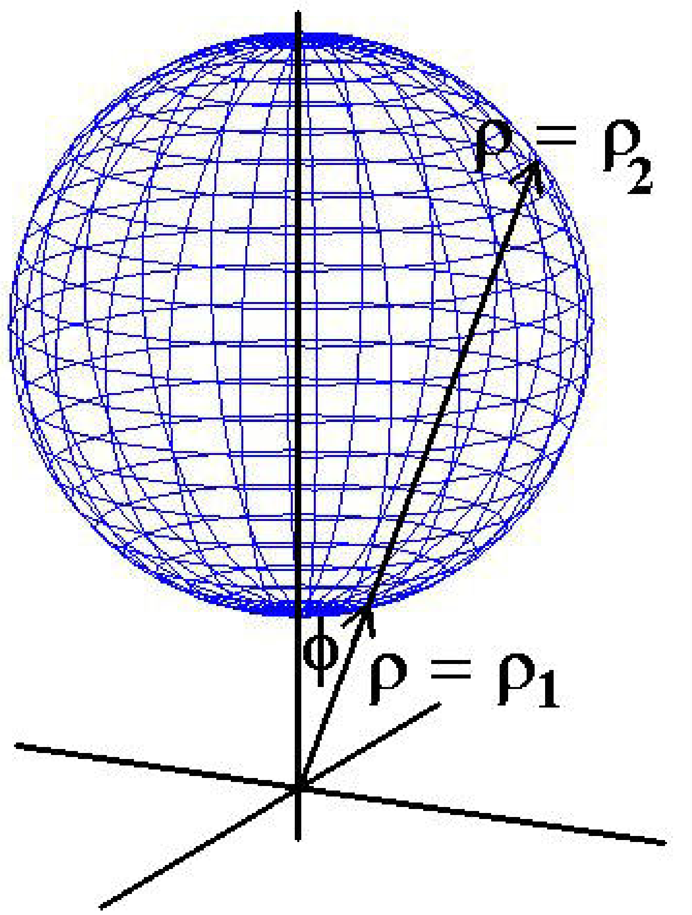
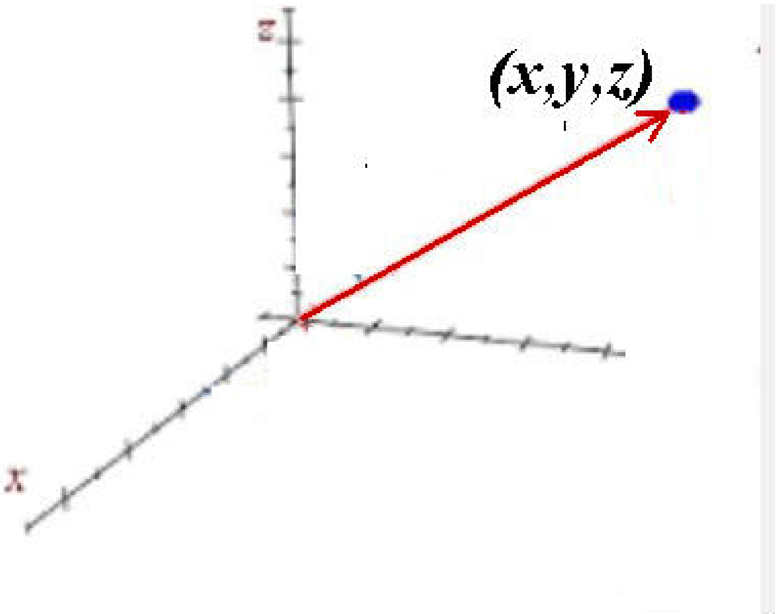
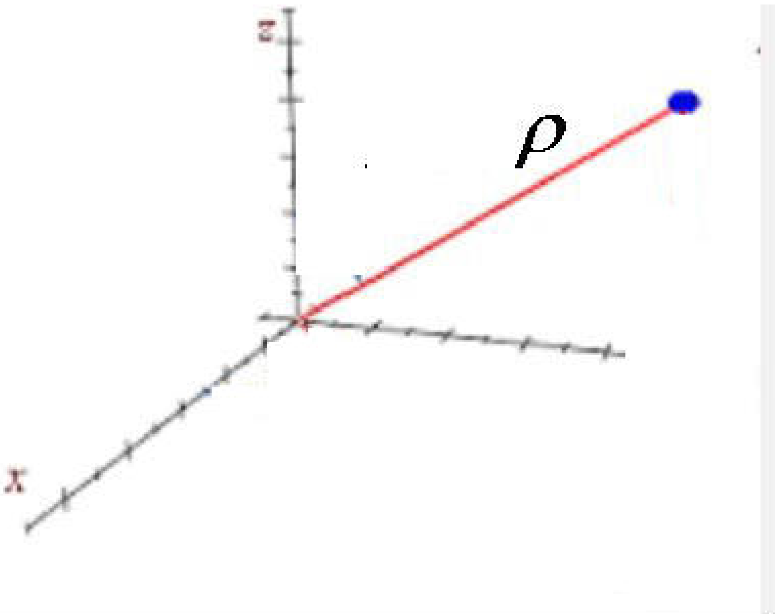
Spherical Coordinates



$$\text{Length of Position Vector} = \sqrt{x^2 + y^2 + z^2}$$



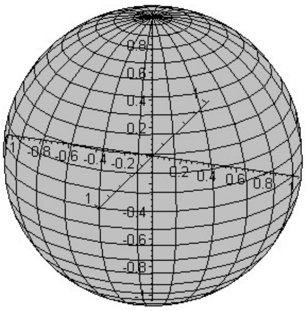
$$\rho = \sqrt{x^2 + y^2 + z^2}$$



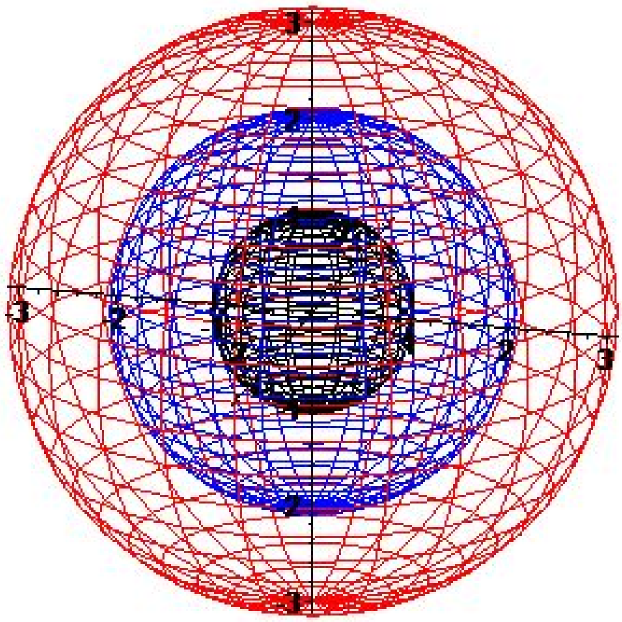
$$x^2 + y^2 + z^2 = 1$$

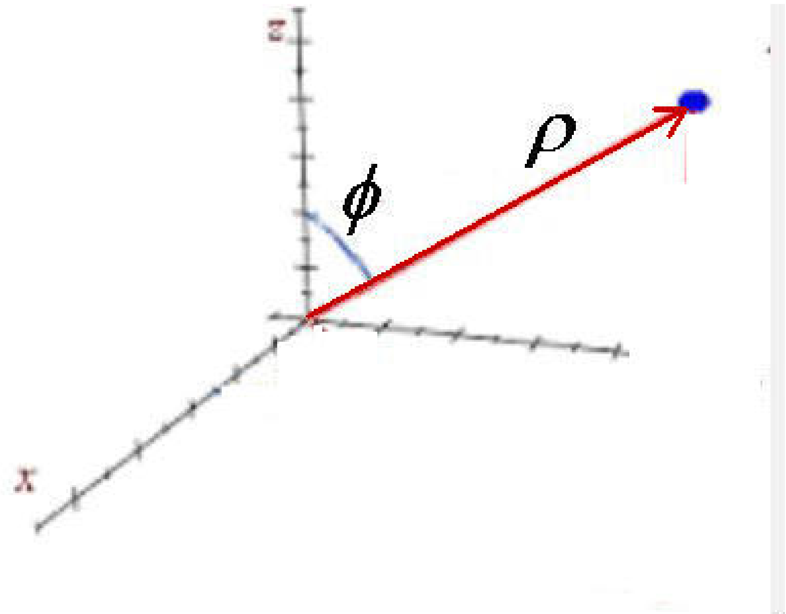
$$\sqrt{x^2 + y^2 + z^2} = 1$$

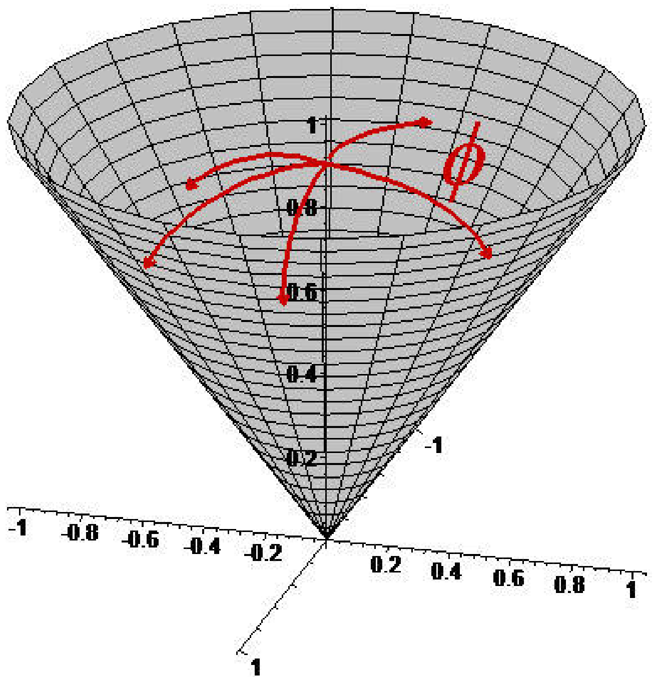
$$\rho = 1$$



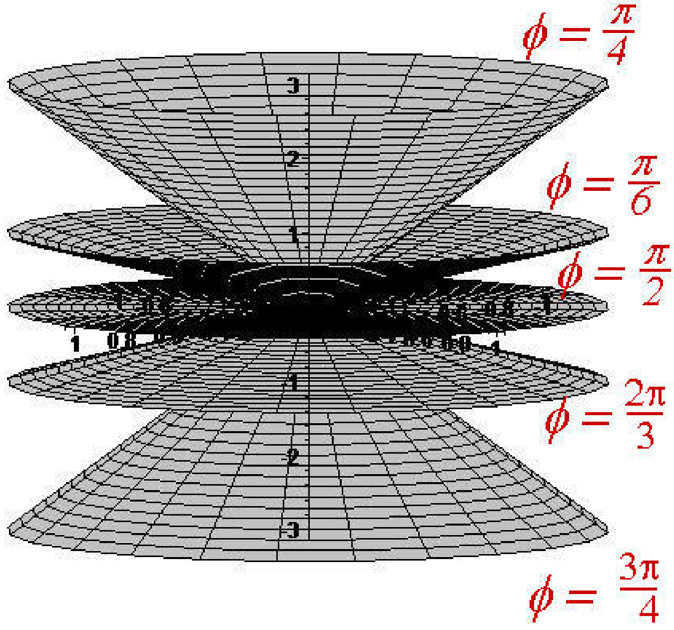
$\rho = constant$

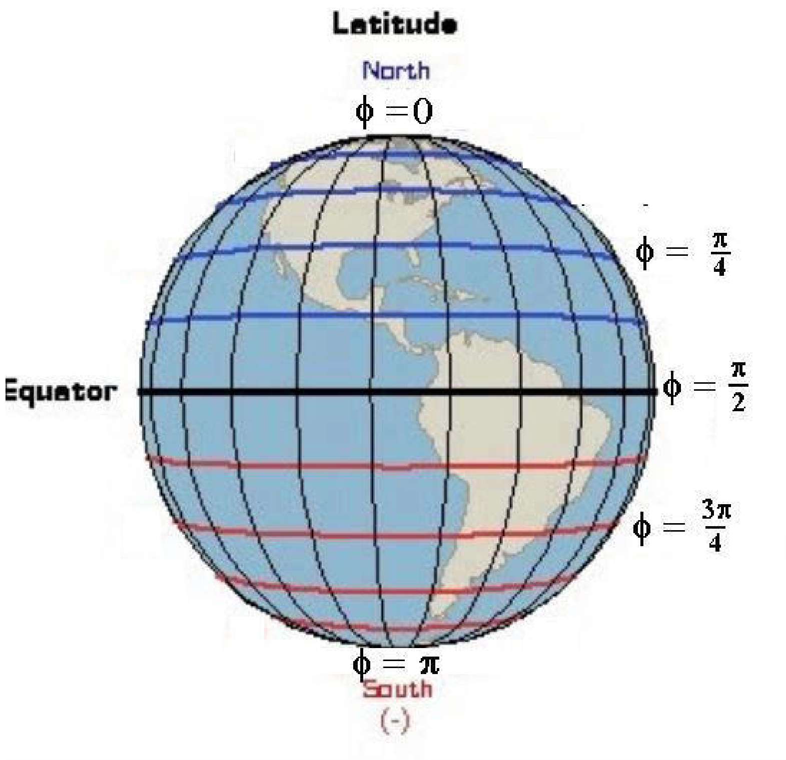



z
x
ϕ
ρ

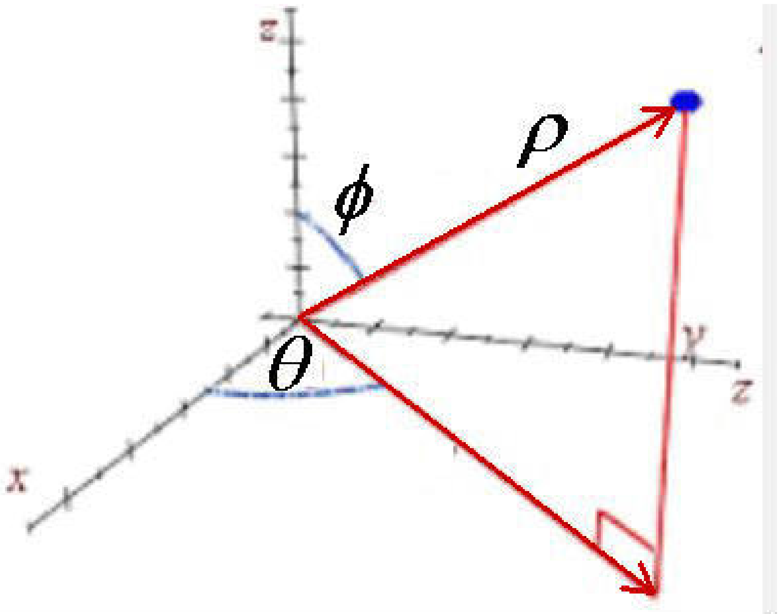


1
0.8
0.6
0.4
0.2
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
1
-1
ϕ

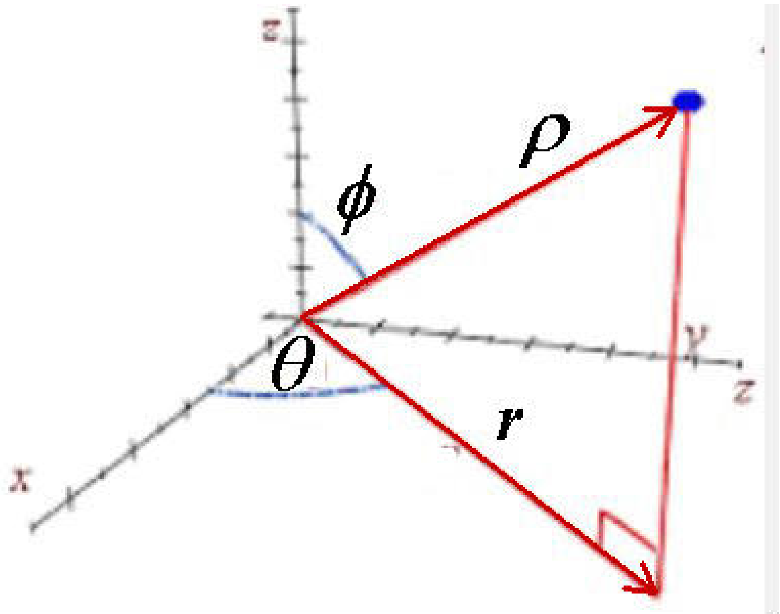

$\phi = \frac{\pi}{4}$
$\phi = \frac{\pi}{6}$
$\phi = \frac{\pi}{2}$
$\phi = \frac{2\pi}{3}$
$\phi = \frac{3\pi}{4}$



Latitude
North
φ = 0
φ = π/4
Equator
φ = π/2
φ = 3π/4
φ = π
South
(-)

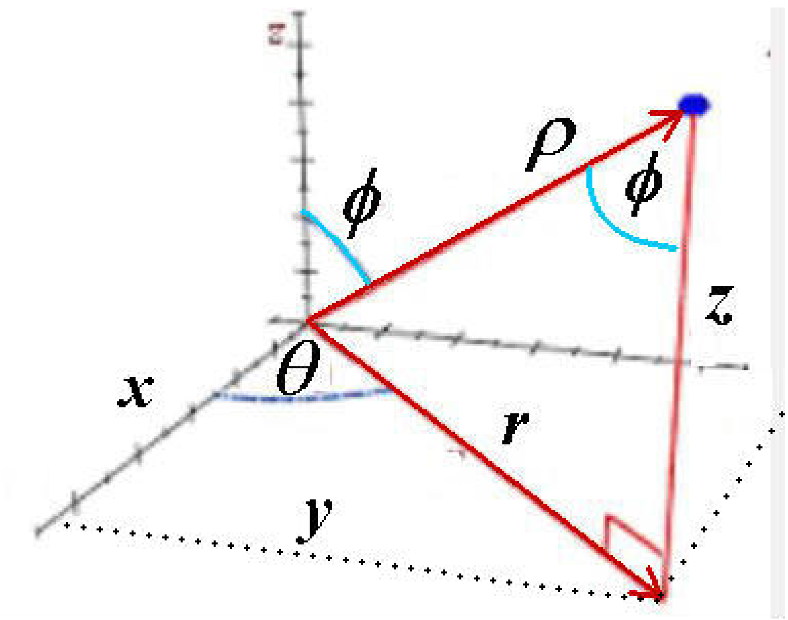


z
ρ
φ
θ
y
z
x

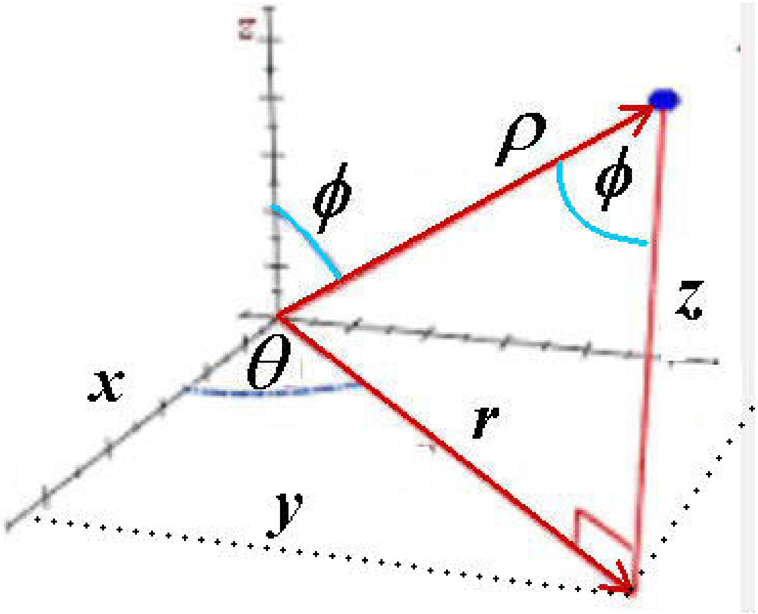


z
ρ
φ
θ
y
z
r
x

$$r = \rho \sin \phi$$

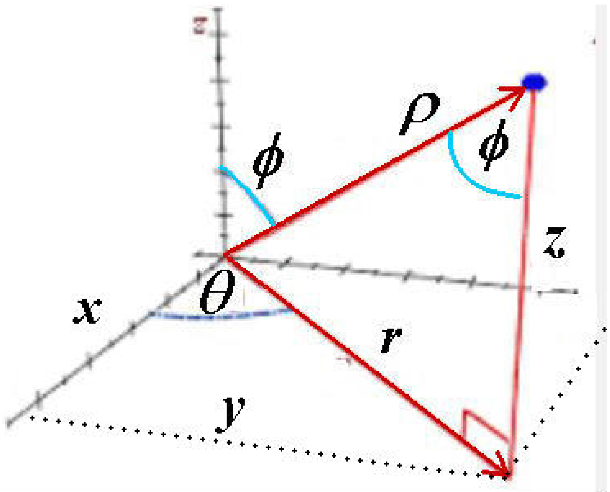


$$r = \rho \sin \phi \qquad z = \rho \cos \phi$$



$$r = \rho \sin \phi \qquad z = \rho \cos \phi$$

$$x = r \cos \theta = \rho \cos \theta \sin \phi \quad y = r \sin \theta = \rho \sin \theta \sin \phi$$



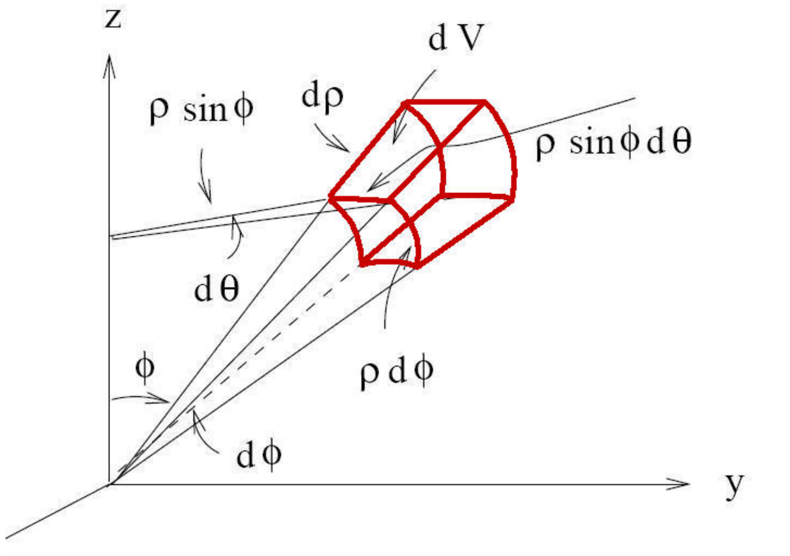
$$
\begin{aligned}
x &= \rho \cos\theta \sin\phi \\
y &= \rho \sin\theta \sin\phi \\
z &= \rho \cos\phi
\end{aligned}
$$

$$\iiint_T \frac{x^2y^3}{x^2+y^2+z^2}\,dV$$

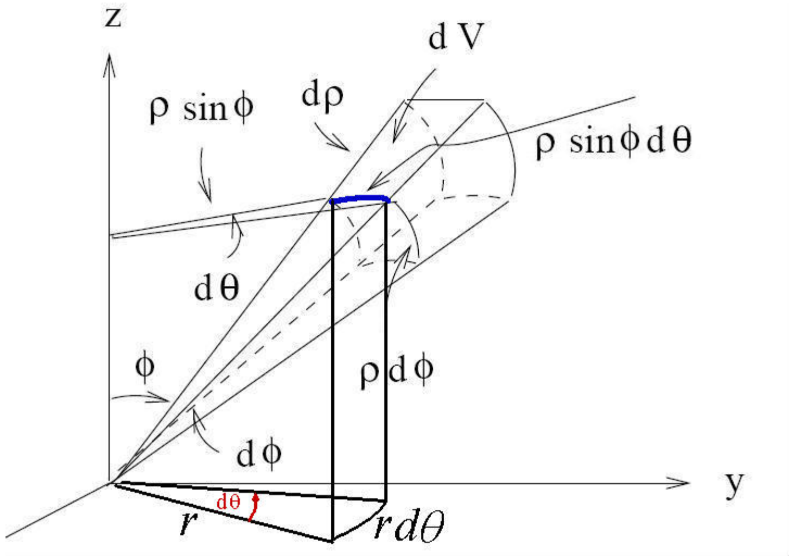
Convert to spherical coordinates:

$$\iiint_T \frac{(\rho\cos\theta\sin\phi)^2(\rho\sin\theta\sin\phi)^3}{\rho^2}\,dV$$

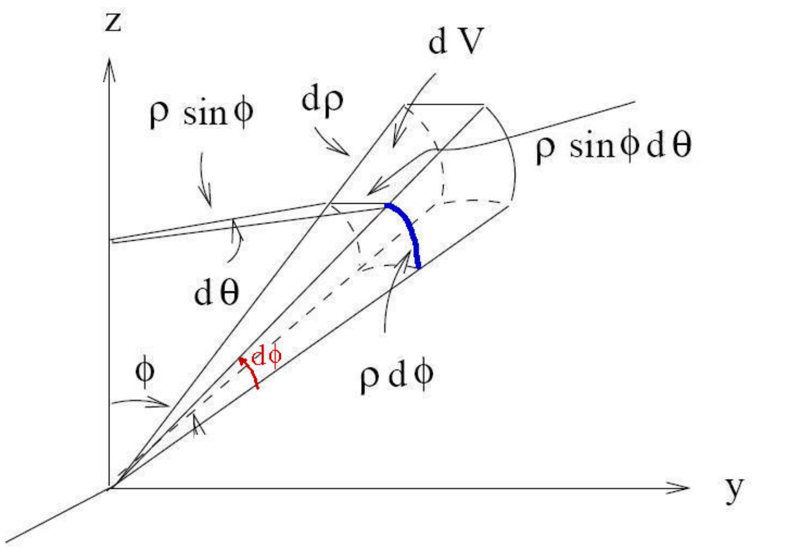
dV is the volume of a *spherical volume element*



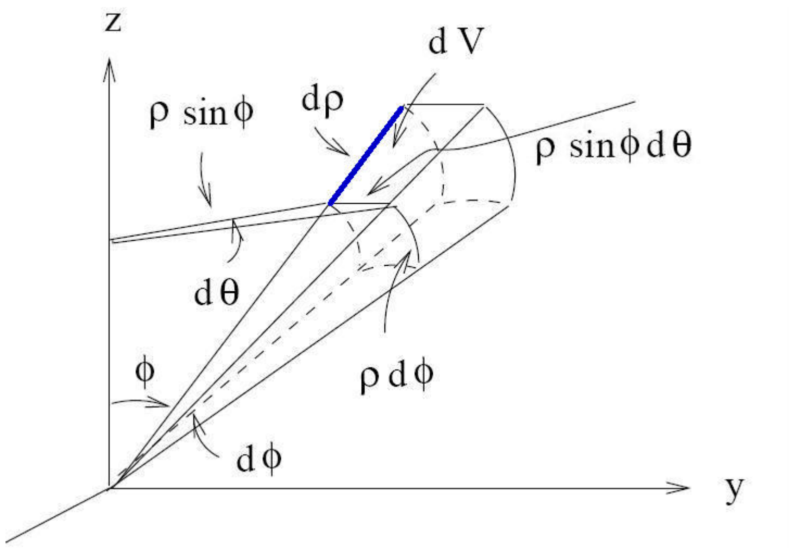
Changing θ by an amount $d\theta$ produces an arc of length $r\,d\theta = \rho \sin\phi\, d\theta$



Changing ϕ by an amount $d\phi$ produces an arc of length $\rho\, d\phi$

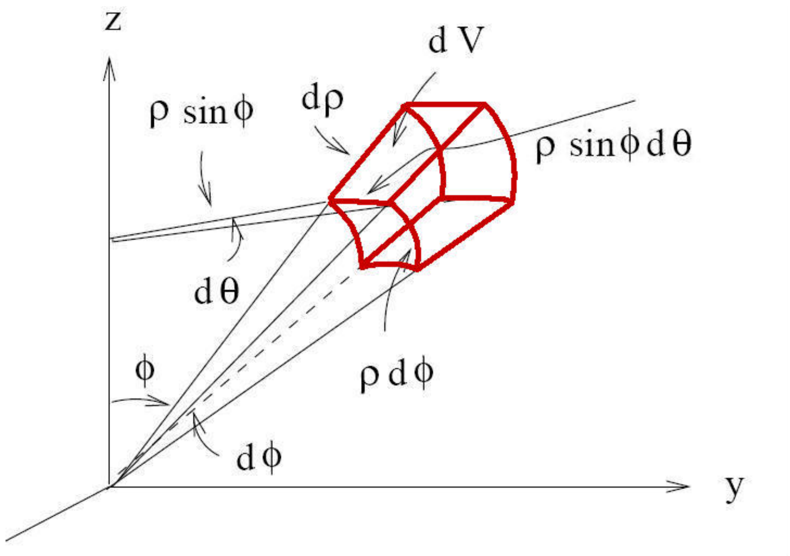


If ρ is changed by an amount $d\rho$, we get the length of another arc along the spherical volume element.

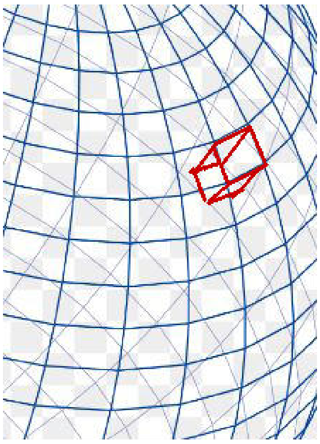


Multiply these edges together to get the volume dV

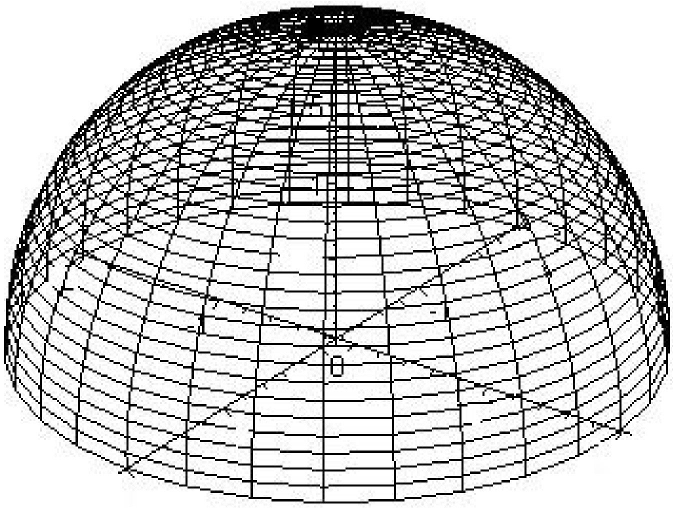
$$dV = (\rho \sin\phi \, d\theta)(\rho \, d\phi)(d\rho) = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



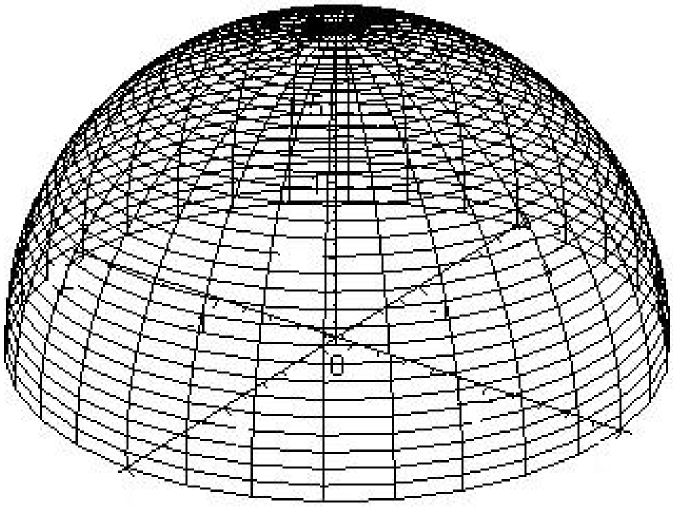
$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



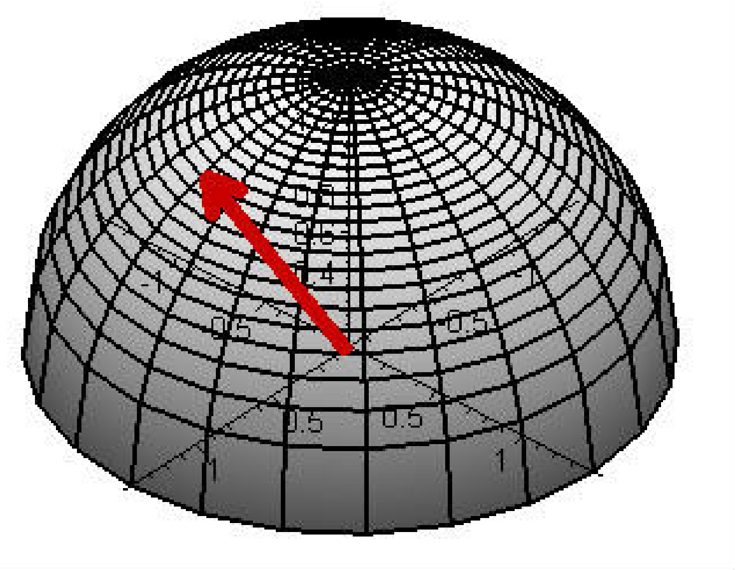
Let $\mathcal{H}$ be the hemispherical region above the xy plane by inside the sphere $x^2 + y^2 + z^2 = 1$.
Calculate $\text{vol}(\mathcal{H}) = \iiint_{\mathcal{H}} 1\, dV$



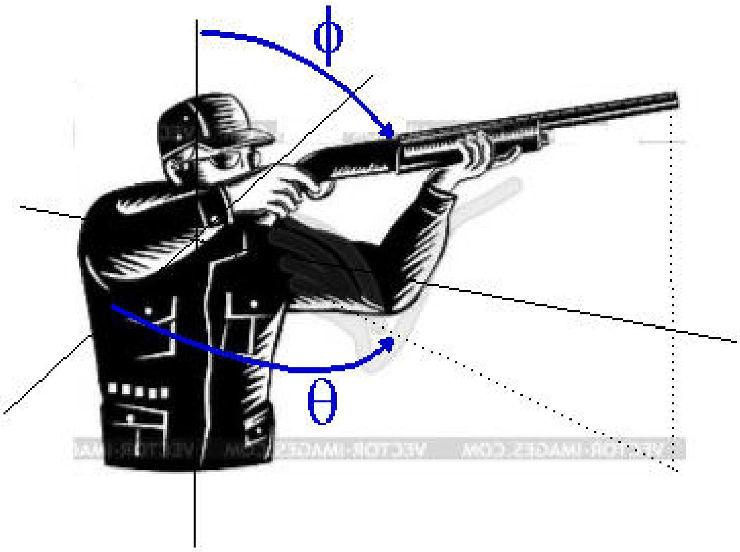
$$\mathrm{vol}(\mathcal{H}) = \iiint 1\,\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



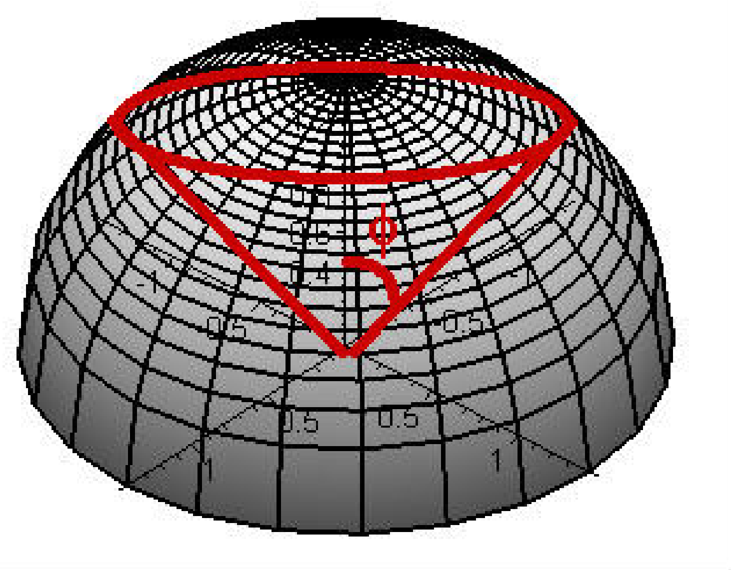
$$\mathrm{vol}(\mathcal{H}) = \iiint_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



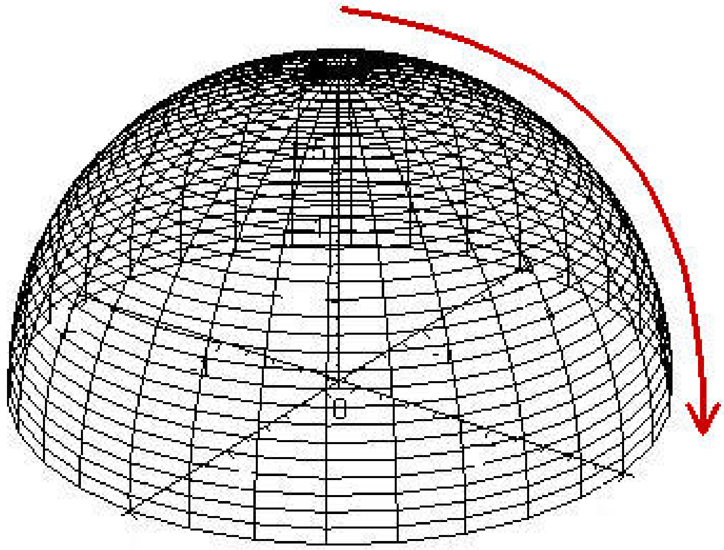
Holding θ and ϕ fixed but varying ρ



$$\text{vol}(\mathcal{H}) = \int\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

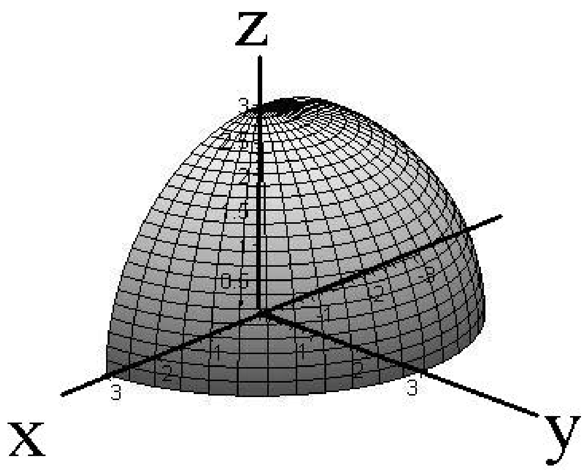


$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



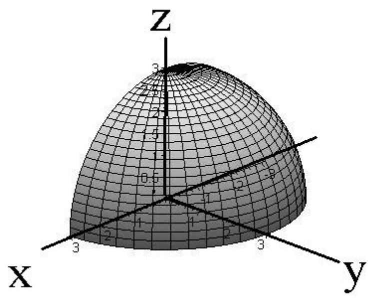
$$
\begin{aligned}
\operatorname{vol}(\mathcal{H}) &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\pi/2}\int_0^{2\pi} \frac{1}{3} \sin\phi \, d\theta \, d\phi \\
&= \int_0^{\pi/2} \frac{2\pi}{3} \sin\phi \, d\phi \\
&= \frac{2\pi}{3}
\end{aligned}
$$

Let Q be the quarter sphere of radius 1 where $y,\, z \geq 0$. Find $\overline{y}$

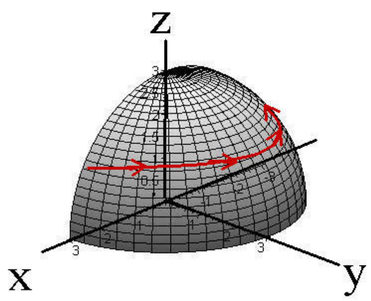


$$\mathrm{Vol}(Q) = \frac{\pi}{3}$$

$$\overline{y} = \frac{1}{\mathrm{Vol}(Q)} \iiint_Q y\, dV = \frac{3}{\pi} \iiint_Q y\, dV$$



$$\begin{aligned}\overline{y} &= \frac{3}{\pi}\iiint_Q y\,dV \\ &= \frac{3}{\pi}\int\int\int(\rho\sin\theta\sin\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \frac{3}{\pi}\int_0^{\pi/2}\int_0^{\pi}\int_0^1 \rho^3\sin\theta\sin^2\phi\,d\rho\,d\theta\,d\phi\end{aligned}$$



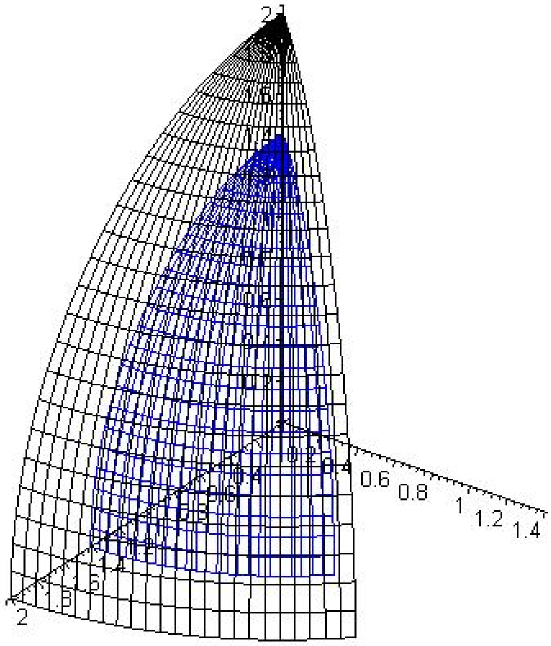
$$\begin{aligned}
\overline{y} &= \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi} \int_0^{1} \rho^3 \sin\theta \sin^2\phi \, d\rho \, d\theta \, d\phi \\
&= \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi} \frac{1}{4} \sin\theta \sin^2\phi \, d\theta \, d\phi
\end{aligned}$$

$$
\begin{aligned}
\overline{y} &= \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi} \int_0^{1} \rho^3 \sin\theta \sin^2\phi \, d\rho \, d\theta \, d\phi \\
&= \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi} \frac{1}{4} \sin\theta \sin^2\phi \, d\theta \, d\phi \\
&= \frac{3}{4\pi} \int_0^{\pi/2} \Big[-\cos\theta \sin^2\phi \Big]_{\theta=0}^{\pi} \, d\phi \\
&= \frac{3}{4\pi} \int_0^{\pi/2} 2\sin^2\phi \, d\phi
\end{aligned}
$$

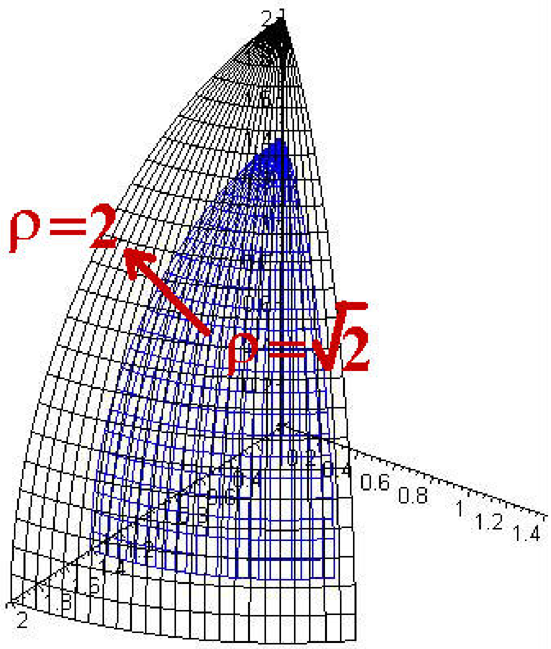
$$
\begin{aligned}
\overline{y} &= \frac{3}{\pi}\int_0^{\pi/2}\int_0^{\pi}\int_0^{1}\rho^3\sin\theta\sin^2\phi\,d\rho\,d\theta\,d\phi \\
&= \frac{3}{\pi}\int_0^{\pi/2}\int_0^{\pi}\frac{1}{4}\sin\theta\sin^2\phi\,d\theta\,d\phi \\
&= \frac{3}{4\pi}\int_0^{\pi/2}\Big[-\cos\theta\sin^2\phi\Big]_{\theta=0}^{\pi}\,d\phi \\
&= \frac{3}{4\pi}\int_0^{1}(1-\cos 2\phi)\,d\phi \\
&= \frac{3}{8}
\end{aligned}
$$

Example:

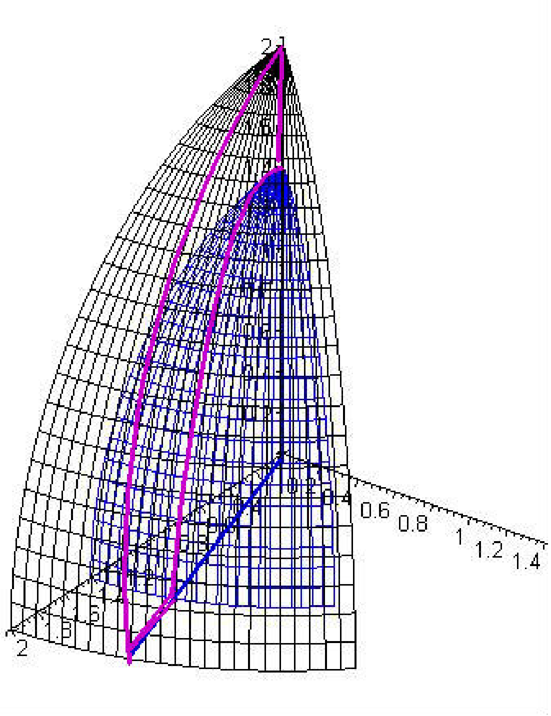
Let T be the region in the first octant bounded by $y = 0$, $y = x$, $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 + z^2 = 4$. Calculate $\iiint_T x \, dV$



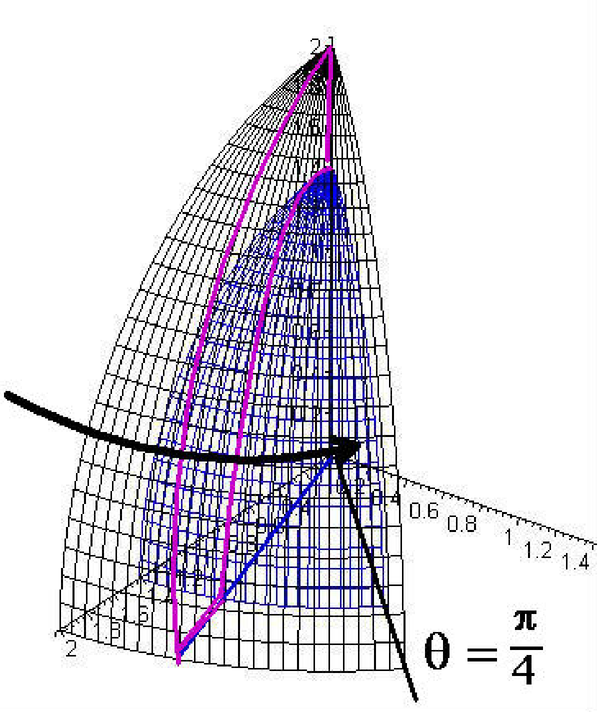
$$\iiint_T x\,dV = \iiint_{\sqrt{2}}^{2} (\rho\cos\theta\sin\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



$$\iiint_T x\,dV = \iint_0^{\pi/2}\int_{\sqrt{2}}^{2} \rho^3 \cos\theta \sin^2\phi \,d\rho\,d\phi\,d\theta$$



$$\iiint_T x\,dV = \int_0^{\pi/4}\int_0^{\pi/2}\int_{\sqrt{2}}^{2} \rho^3 \cos\theta \sin^2\phi \, d\rho\, d\phi\, d\theta$$



$$\iiint_T x\,dV = \int_0^{\pi/4}\int_0^{\pi/2}\int_{\sqrt{2}}^{2} \rho^3 \cos\theta \sin^2\phi \, d\rho\, d\phi\, d\theta$$

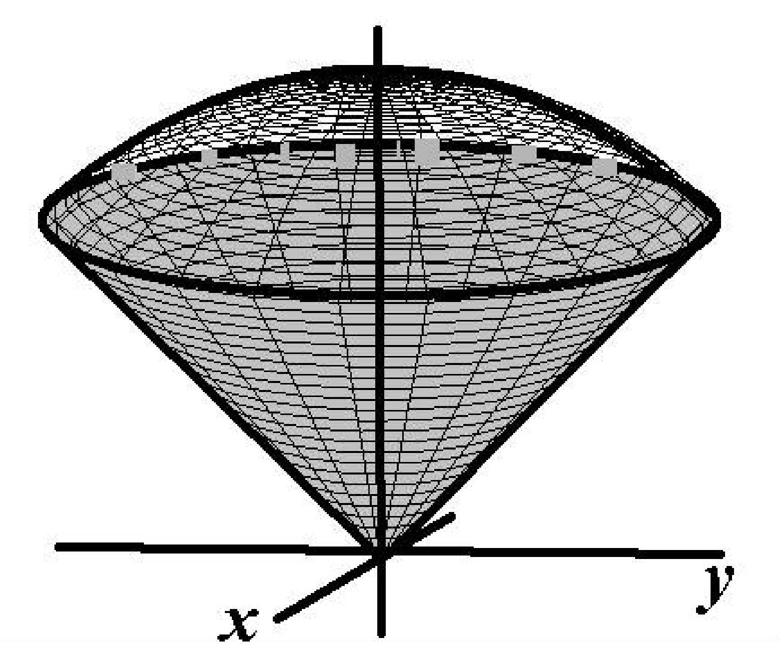
$$= \int_0^{\pi/4}\int_0^{\pi/2} 3\cos\theta \sin^2\phi \, d\phi\, d\theta$$

$$\iiint_T x\,dV = \int_0^{\pi/4}\int_0^{\pi/2}\int_{\sqrt{2}}^{2} \rho^3 \cos\theta \sin^2\phi \,d\rho\,d\phi\,d\theta$$
$$= \int_0^{\pi/4}\int_0^{\pi/2} 3\cos\theta\sin^2\phi\,d\phi\,d\theta$$
$$= \left(\int_0^{\pi/4} 3\cos\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^2\phi\,d\phi\right)$$

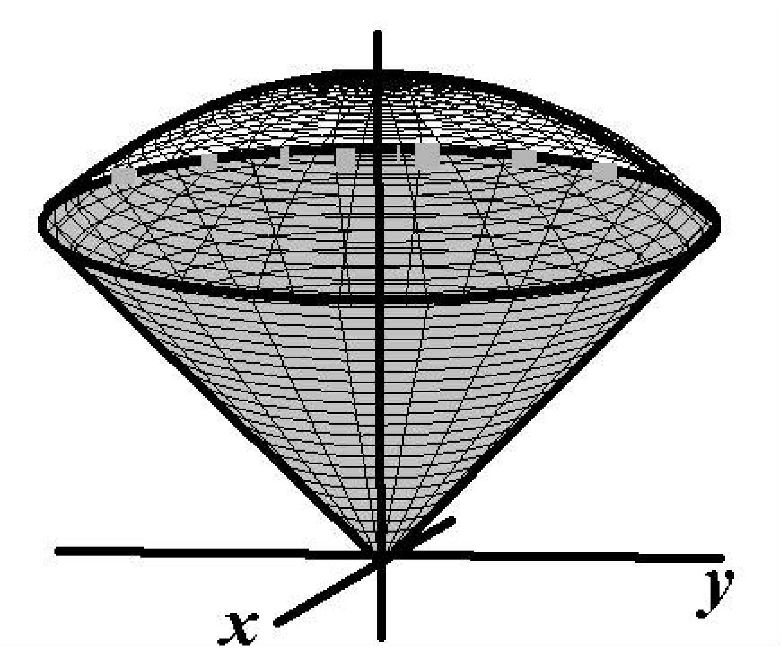
$$
\begin{aligned}
\iiint_T x\,dV &= \int_0^{\pi/4}\int_0^{\pi/2}\int_{\sqrt{2}}^{2} \rho^3 \cos\theta \sin^2\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi/4}\int_0^{\pi/2} 3\cos\theta \sin^2\phi \, d\phi\, d\theta \\
&= \left(\int_0^{\pi/4} 3\cos\theta\, d\theta\right)\left(\int_0^{\pi/2} \sin^2\phi\, d\phi\right) \\
&= \frac{3\sqrt{2}}{8}\pi
\end{aligned}
$$

Example:

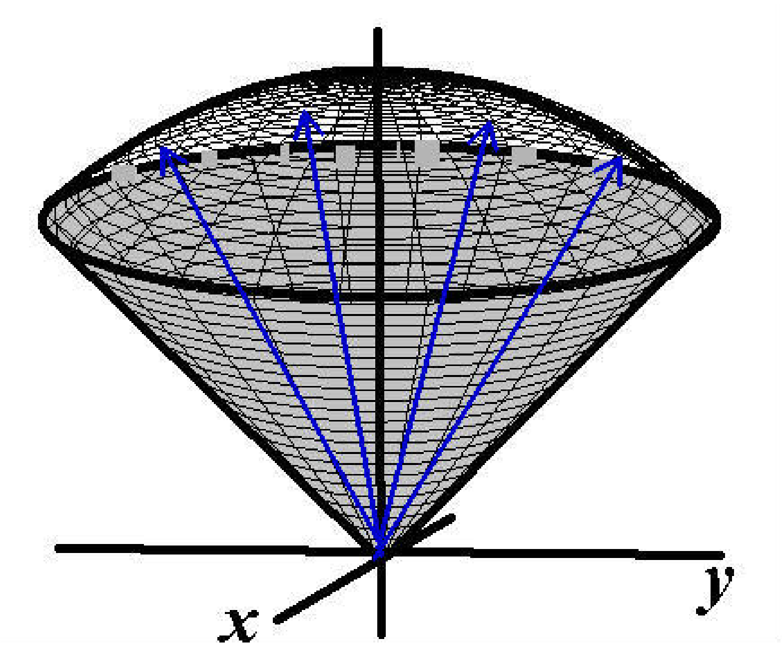
Find the volume of the region Q that is inside the sphere $x^2 + y^2 + z^2 = 2$ but above $z = \sqrt{x^2 + y^2}$.



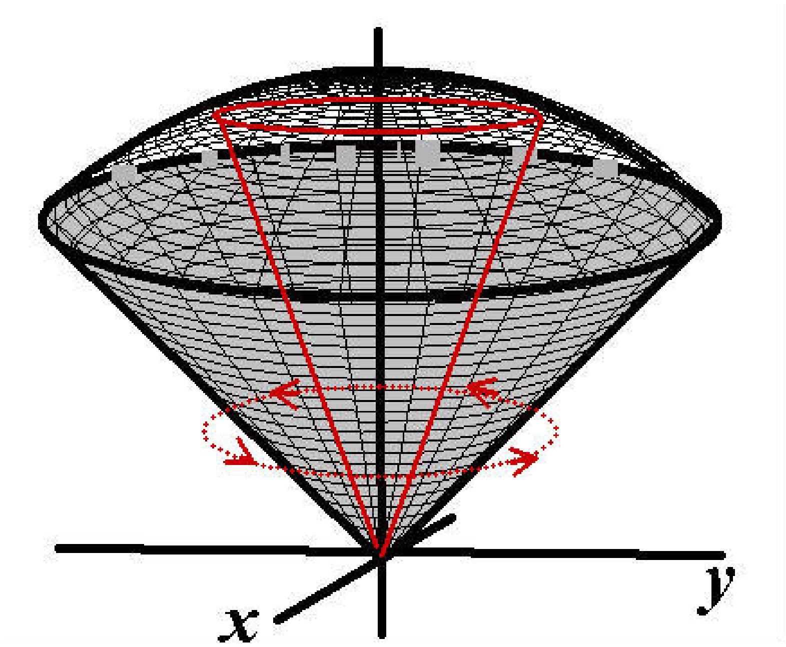
$$\iiint_Q 1\,dV = \iiint_Q \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$



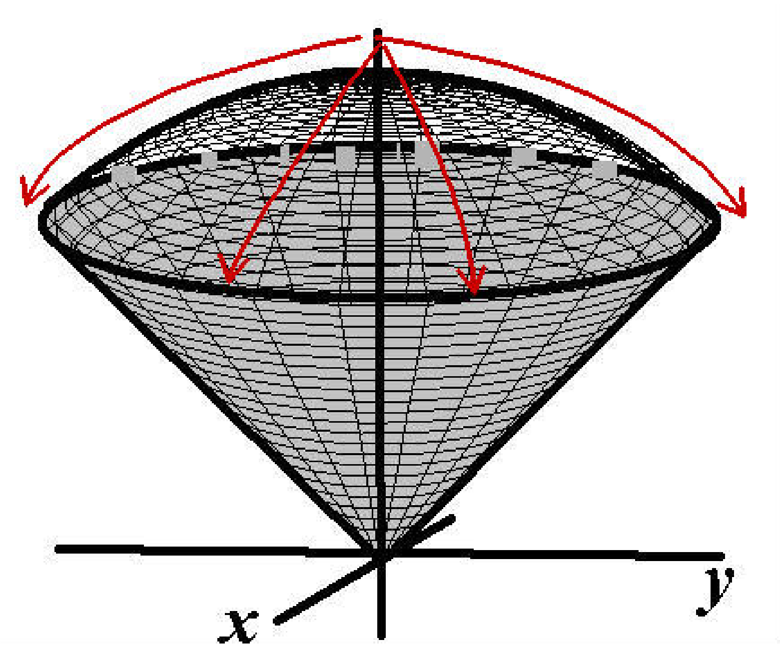
$$\iiint_Q 1\, dV = \iiint_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



$$\iiint_Q 1\, dV = \iint_0^{2\pi} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

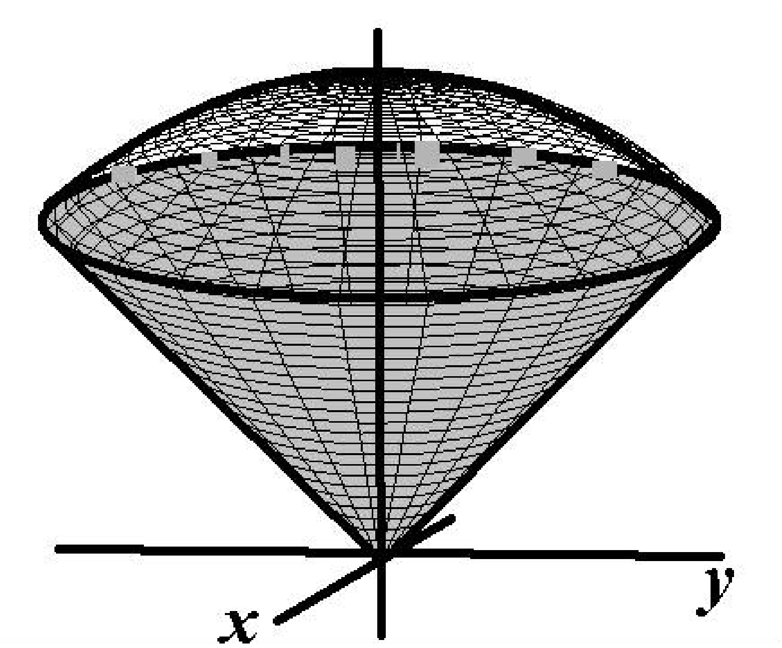


$$\iiint_Q 1\, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

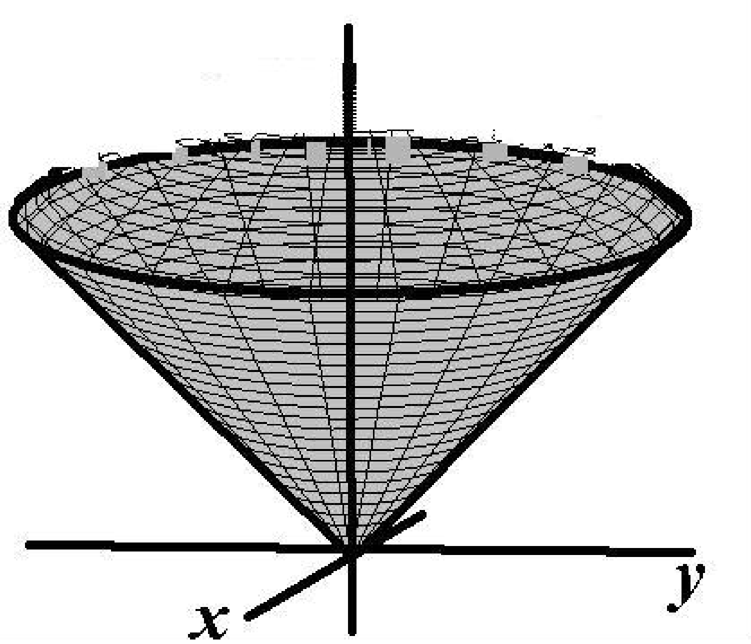


$$\iiint_Q 1\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sqrt{2}} \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \frac{4\pi}{3}(\sqrt{2}-1)$$

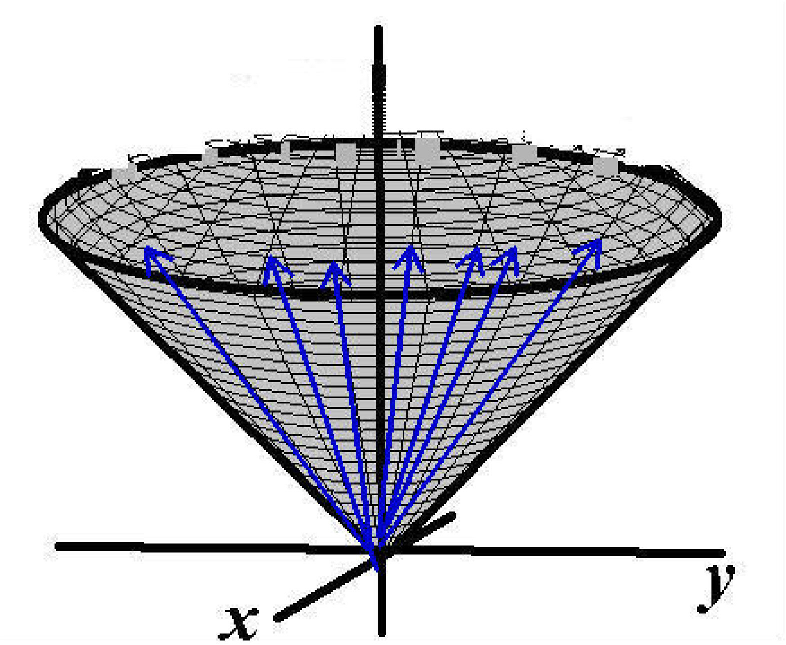
The cone and the sphere intersect at $z = 1$



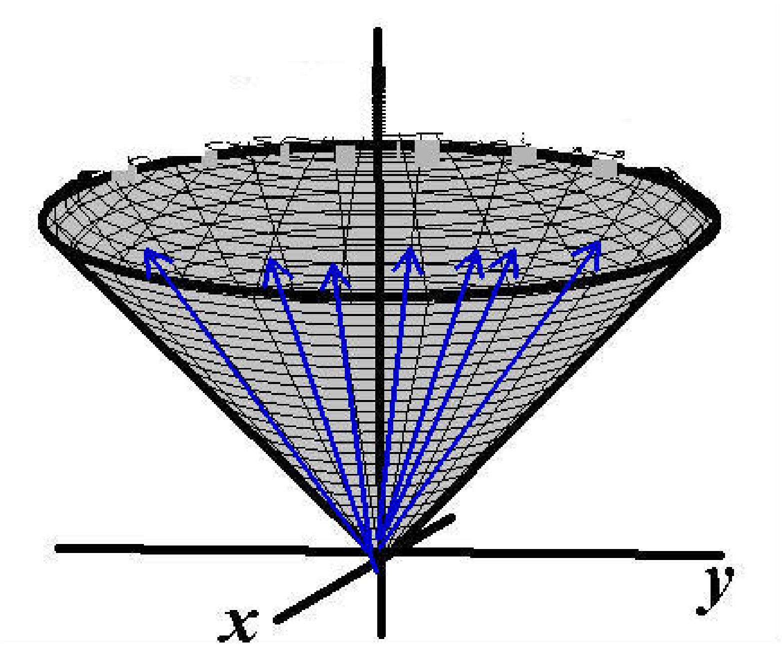
Problem Modification: Find the volume of the portion of the cone below $z = 1$



$$\text{Volume} = \iiint \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



The largest ρ value occurs on the cone $z = 1$

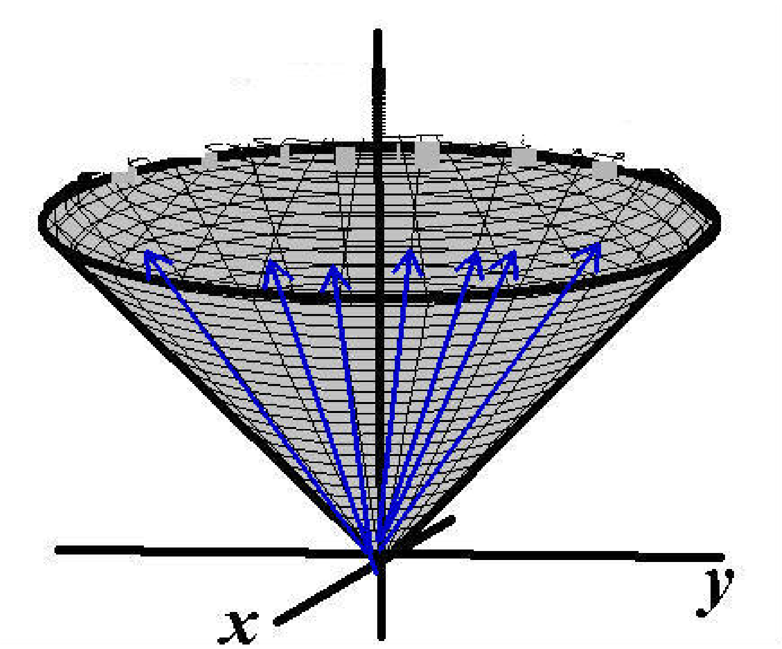


$$z = 1$$

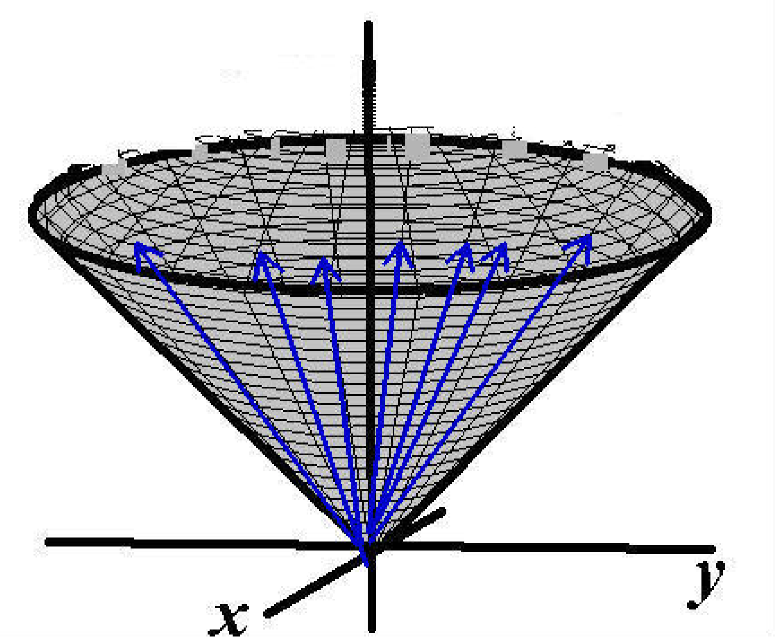
$$\rho \cos \phi = 1$$

$$\rho = \frac{1}{\cos \phi} = \sec \phi$$

$$\text{Volume} = \int\int\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



$$\text{Volume} = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



$$
\begin{aligned}
\text{Volume} &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\pi/4} \int_0^{2\pi} \frac{1}{3} \sec^3\phi \sin\phi \, d\theta \, d\phi
\end{aligned}
$$

$$
\begin{aligned}
\text{Volume} &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\pi/4} \int_0^{2\pi} \frac{1}{3} \sec^3\phi \sin\phi \, d\theta \, d\phi \\
&= \int_0^{\pi/4} \frac{2\pi}{3} \sec^3\phi \sin\phi \, d\phi
\end{aligned}
$$

$$
\begin{aligned}
\sec^3\phi\sin\phi &= \sec^2\phi\sec\phi\sin\phi \\
&= \sec^2\phi\cdot\frac{1}{\cos\phi}\cdot\sin\phi \\
&= \sec^2\phi\cdot\tan\phi
\end{aligned}
$$

$$
\begin{aligned}
\text{Volume} &= \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\pi/4}\int_0^{2\pi} \frac{1}{3}\sec^3\phi \sin\phi \, d\theta \, d\phi \\
&= \int_0^{\pi/4} \frac{2\pi}{3}\sec^3\phi \sin\phi \, d\phi \\
&= \frac{2\pi}{3}\int_0^{\pi/4} \tan\phi \sec^2\phi \, d\phi \\
&= \frac{2\pi}{3}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} = \frac{\pi}{3}
\end{aligned}
$$

In spherical coordinates:

$$\text{Volume} = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

In cylindrical coordinates:

$$\text{Volume} = \int_0^{1} \int_0^{2\pi} \int_r^{1} r \, dz \, d\theta \, dr$$