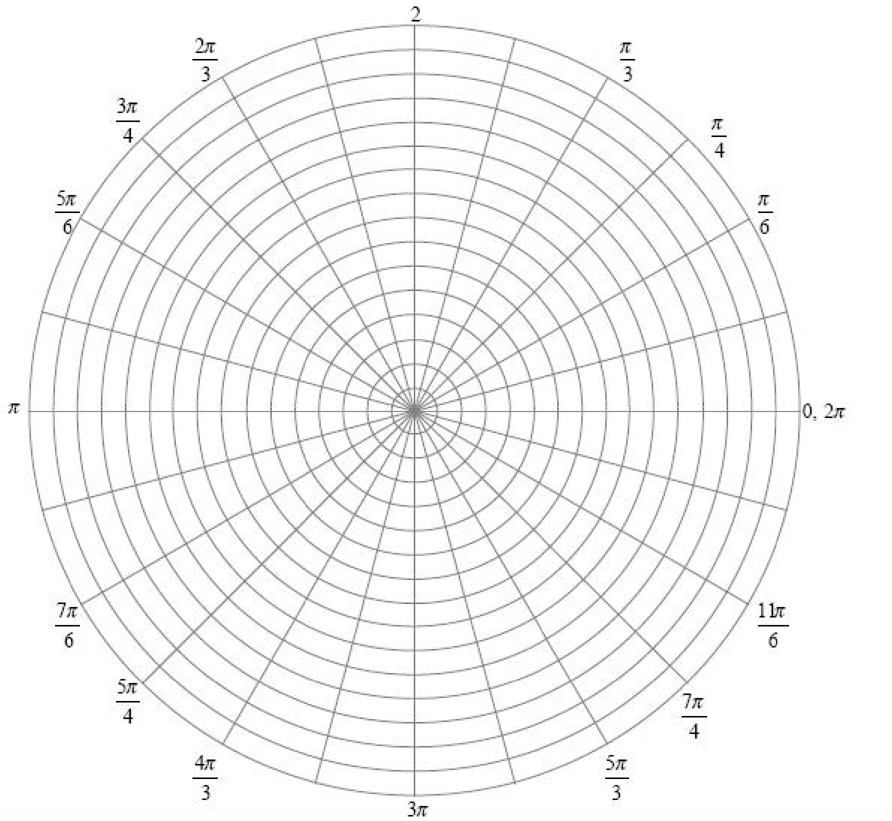
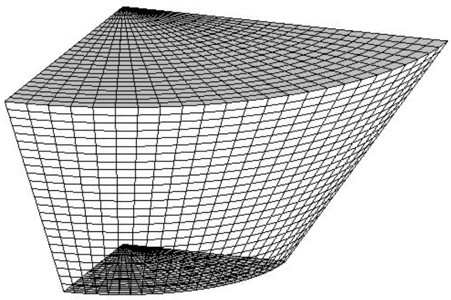
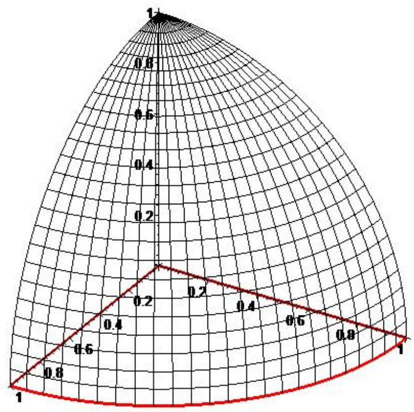
Change of Variables for Multiple Integrals






1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
0.2
0.4
0.6
0.8
1

Polar Coordinates

Input values of r and θ

$$x = r\cos\theta \qquad y = r\sin\theta$$

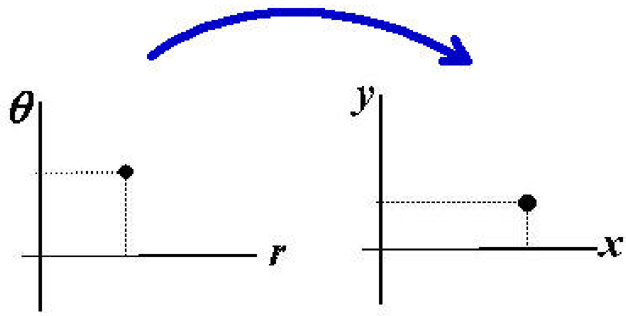
Output values of x and y

Polar Coordinates

Input point $(r,\ \theta)$

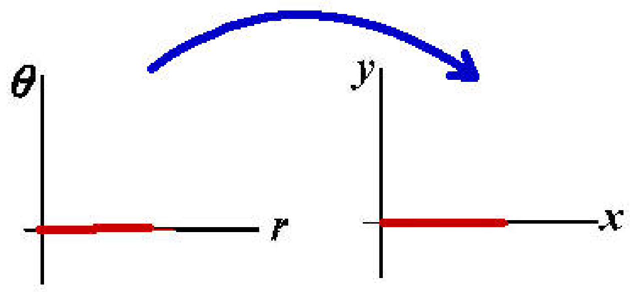
$$x = r\cos\theta \qquad y = r\sin\theta$$

Output point $(x,\ y)$



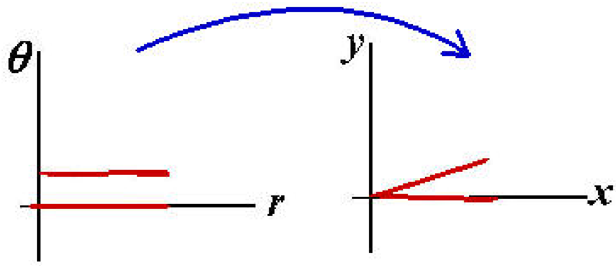
$$x = r\cos\theta \qquad y = r\sin\theta$$

$\theta = 0$. Vary r



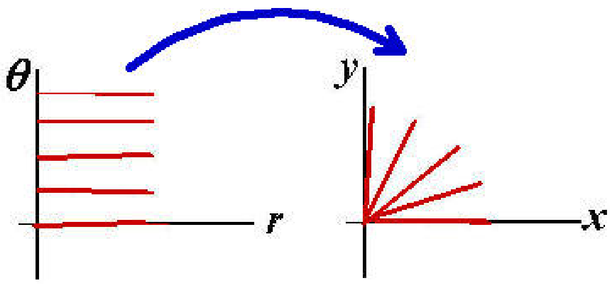
$$x = r\cos\theta \qquad y = r\sin\theta$$

Larger θ. Vary r



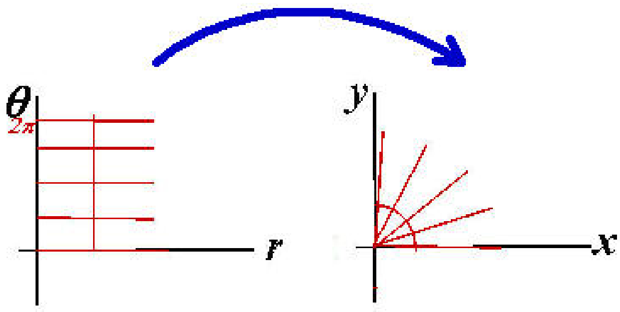
$$x = r\cos\theta \qquad y = r\sin\theta$$

Vary r for different values of θ



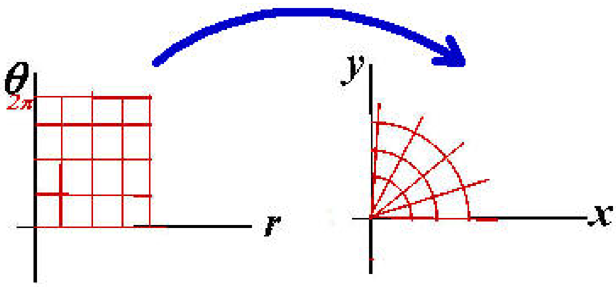
$$x = r\cos\theta \qquad y = r\sin\theta$$

Vary θ for some value of r



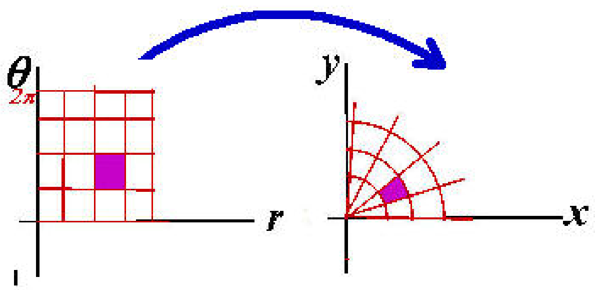
$$x = r\cos\theta \qquad y = r\sin\theta$$

Vary θ for different values of r

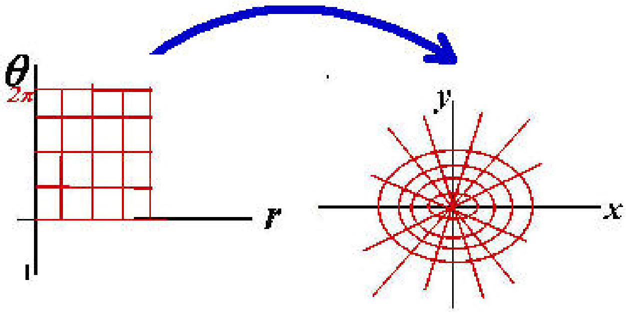


$$x = r\cos\theta \qquad y = r\sin\theta$$

Every rectangular grid section in the $r\theta$ plane is transformed into a polar section in the xy plane.



$$x = 3r\cos\theta \qquad y = 2r\sin\theta$$

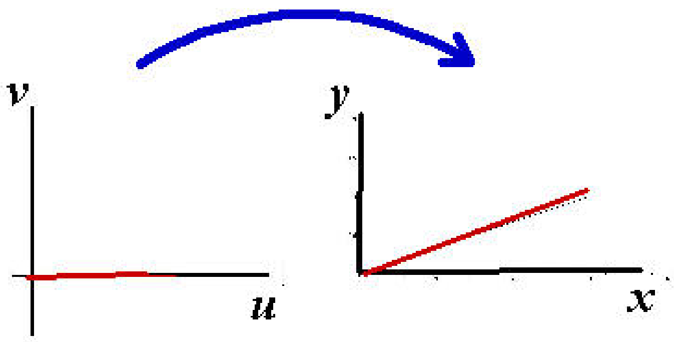


$$x = 3u + v$$
$$y = u + v$$

These equations will transform a rectangular grid in the uv plane to some type of grid in the xy plane.

$$x = 3u + v$$
$$y = u + v$$

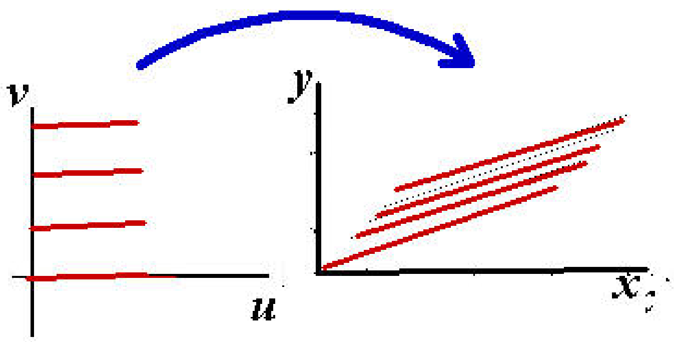
If $v = 0$ then $x = 3u$ and $y = u$ so $y = \frac{1}{3}x$



$$x = 3u + v$$

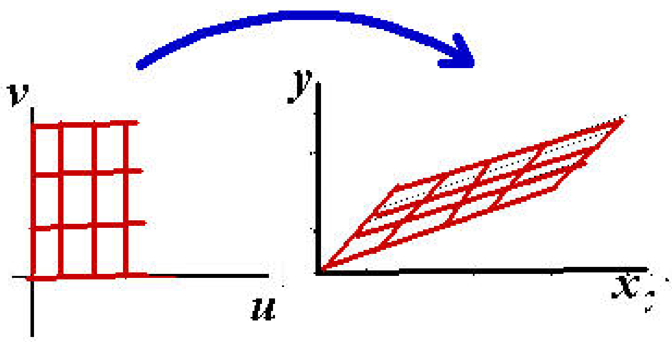
$$y = u + v$$

Graph the line for different values of v



$$x = 3u + v$$
$$y = u + v$$

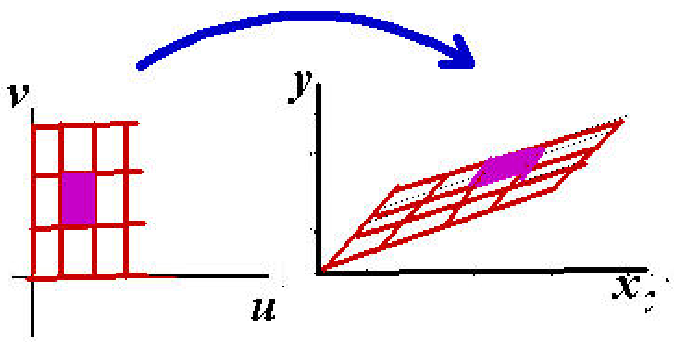
Now draw the lines for different values of u



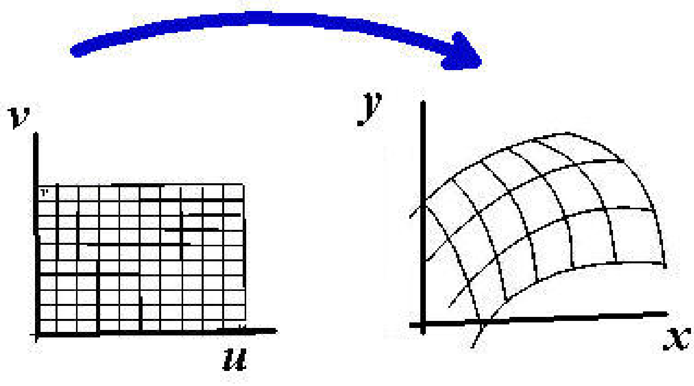
$$x = 3u + v$$

$$y = u + v$$

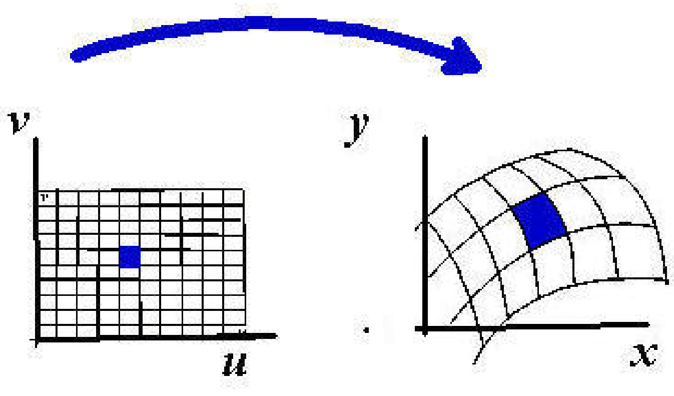
A rectangular grid section in the uv plane is transformed into a parallelogram section in the xy plane.



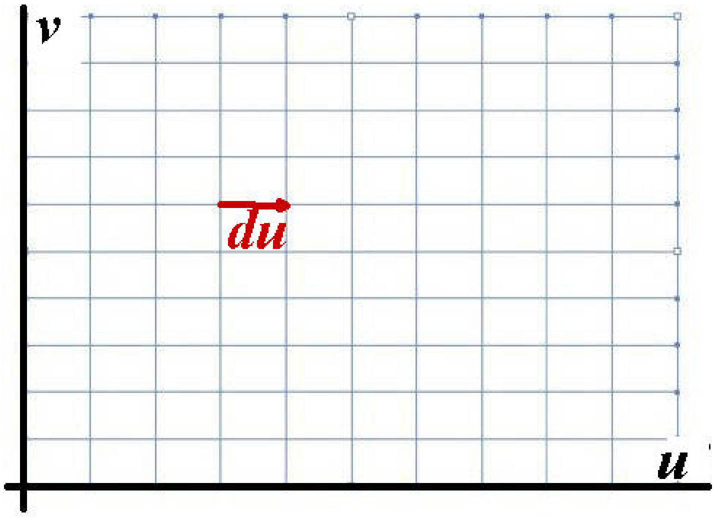
$$x = x(u, v) \qquad y = y(u, v)$$



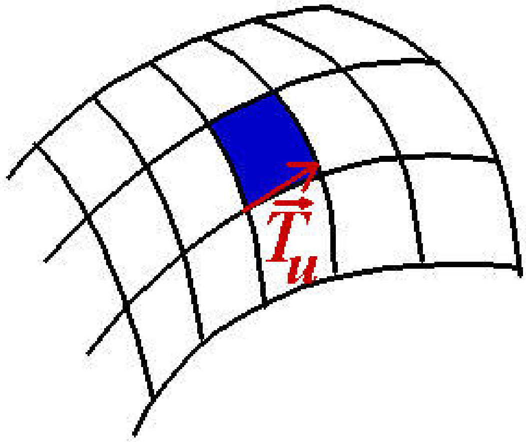
$$x = x(u, v) \qquad y = y(u, v)$$



Change u by an amount du



Let $\vec{\mathbf{T}}_u$ be the tangent vector that approximates the change in the grid in the xy plane.

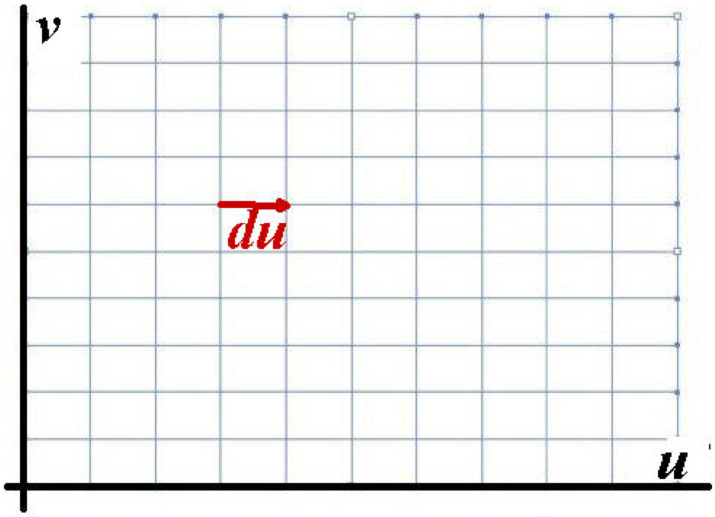


Let $\vec{\mathbf{T}}_u$ be the tangent vector that approximates the change in the grid in the xy plane.

$$\vec{\mathbf{T}}_u = \langle dx,\ dy \rangle$$

$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv \qquad dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv$$

However, if we are only change u in the uv plane then $dv = 0$



$$\vec{\mathbf{T}}_u = \langle dx,\ dy \rangle$$

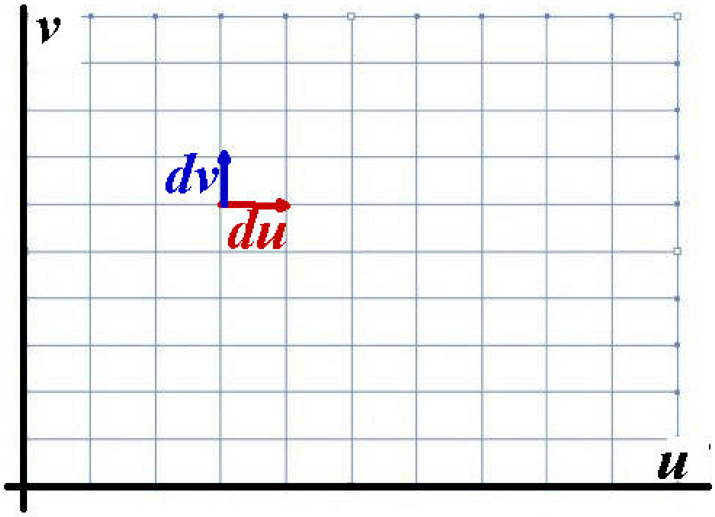
$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv \qquad dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv$$

If $dv = 0$, this simplifies to:

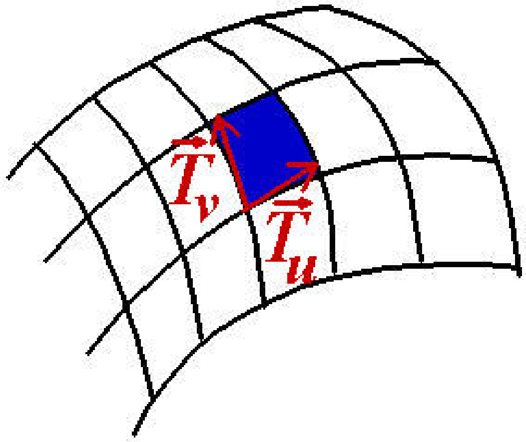
$$dx = \frac{\partial x}{\partial u}\,du \qquad dy = \frac{\partial y}{\partial u}\,du$$

$$\vec{\mathbf{T}}_u = \left\langle \frac{\partial x}{\partial u},\ \frac{\partial y}{\partial u} \right\rangle du$$

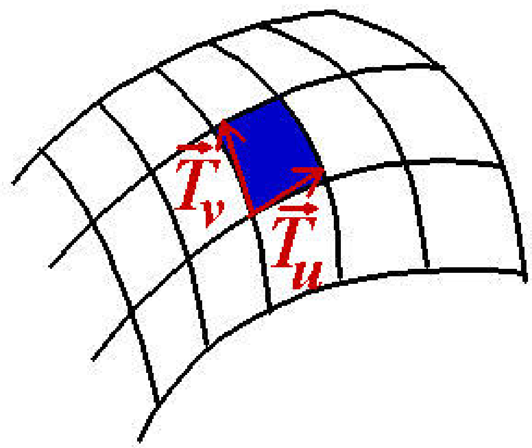
Change v by an amount dv



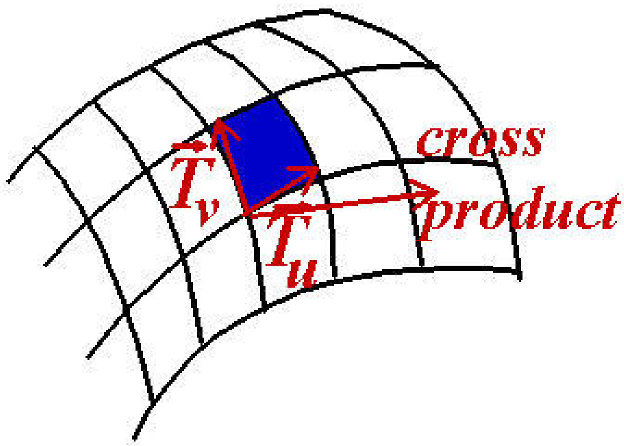
Let $\vec{\mathbf{T}}_v$ be the tangent vector that approximates the change in the grid in the xy plane.



$$\vec{\mathbf{T}}_u = \left\langle \frac{\partial x}{\partial u},\ \frac{\partial y}{\partial u} \right\rangle du \qquad \vec{\mathbf{T}}_v = \left\langle \frac{\partial x}{\partial v},\ \frac{\partial y}{\partial v} \right\rangle dv$$



$$dA = |\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v|$$



$$\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\, dv$$

$$\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\, dv$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{\mathbf{k}}\, du\, dv$$

$$\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\, dv$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{\mathbf{k}}\, du\, dv$$

$$dA = |\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right| du\, dv$$

$$\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\, dv$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{\mathbf{k}}\, du\, dv$$

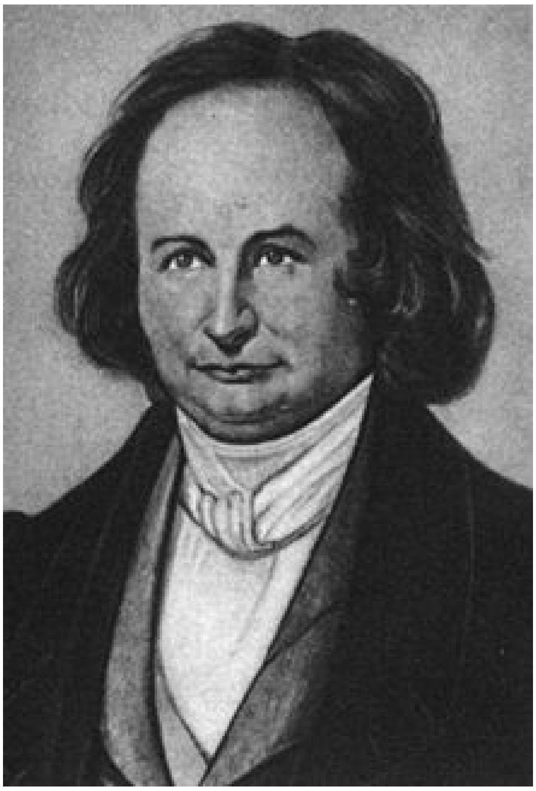
$$dA = |\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right| du\, dv$$

$$= |x_u y_v - y_u x_v|\; du\, dv$$

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

The Jacobian determinant:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

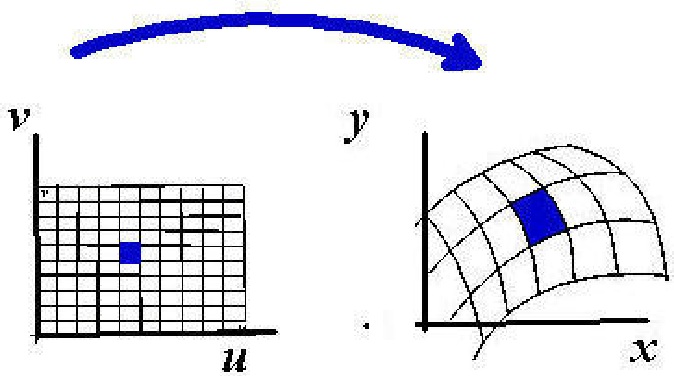


The Jacobian determinant:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

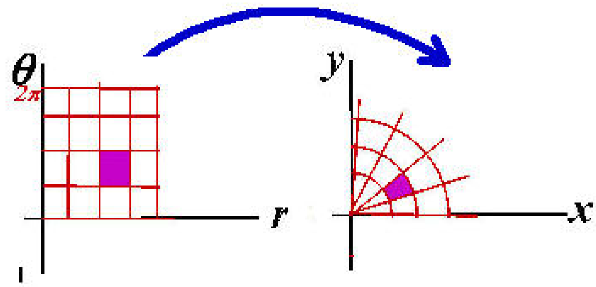
$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$dA = J\,du\,dv$$



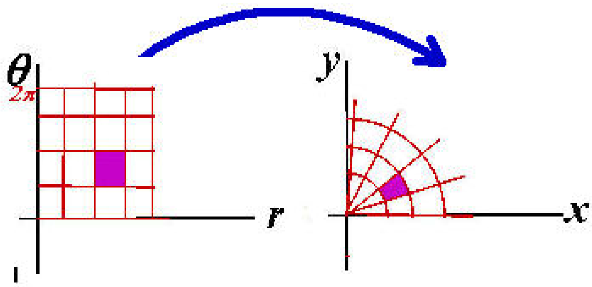
$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dA = J\, d\theta\, dr$$



$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dA = J\,d\theta\,dr = \frac{\partial(x,\,y)}{\partial(r,\theta)}\,d\theta\,dr$$



$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial(x,\,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial(x,\, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

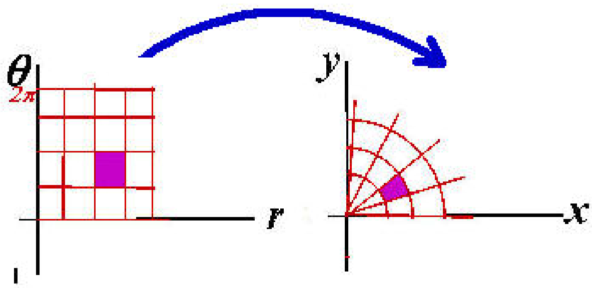
$$\frac{\partial(x,\,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

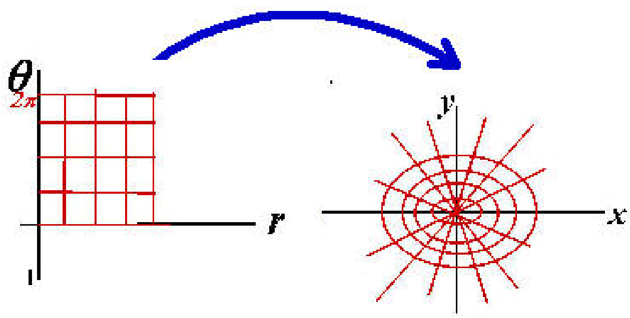
$$= r$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dA = J\,d\theta\,dr = r\,d\theta\,dr$$



$$x = ar\cos\theta \qquad y = br\sin\theta$$



$$x = ar\cos\theta \qquad y = br\sin\theta$$

$$\frac{x}{ar} = \cos\theta \qquad \frac{y}{br} = \sin\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$x = ar\cos\theta \qquad y = br\sin\theta$$

$$\frac{x}{ar} = \cos\theta \qquad \frac{y}{br} = \sin\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{x^2}{(ar)^2} + \frac{y^2}{(br)^2} = 1$$

If $r = 1$ then:

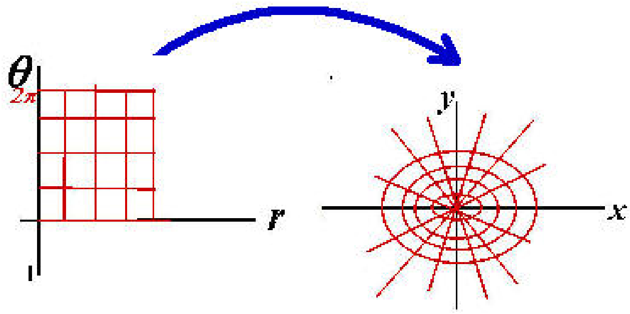
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the equation of an ellipse.

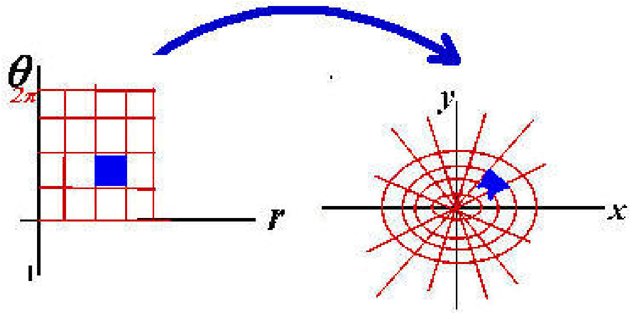
$$x = ar\cos\theta \qquad y = br\sin\theta$$

$$\frac{x^2}{(ar)^2} + \frac{y^2}{(br)^2} = 1$$

If $0 \leq r \leq 1$ then we get several concentric ellipses.



$$dA = \frac{\partial(x,y)}{\partial(r,\theta)}\, d\theta\, dr$$



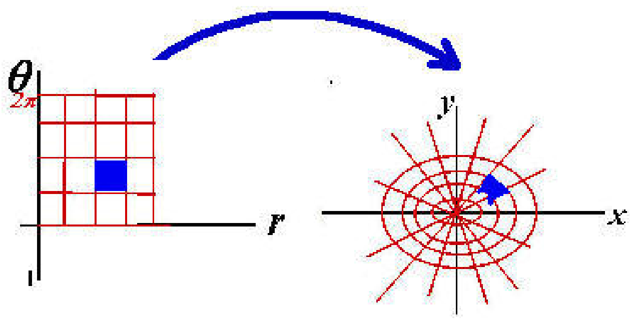
$$x = ar\cos\theta \qquad y = br\sin\theta$$

$$\frac{\partial(x,\, y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

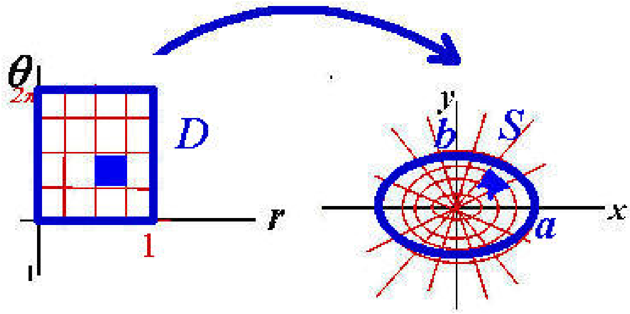
$$= \begin{vmatrix} a\cos\theta & b\sin\theta \\ -ra\sin\theta & rb\cos\theta \end{vmatrix}$$

$$= abr$$

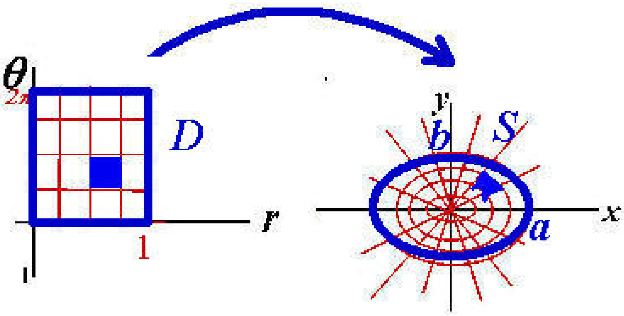
$$dA = \frac{\partial(x,y)}{\partial(r,\theta)}\, d\theta\, dr = abr\, d\theta\, dr$$



$$dA = \frac{\partial(x, y)}{\partial(r, \theta)}\, d\theta\, dr = abr\, d\theta\, dr$$



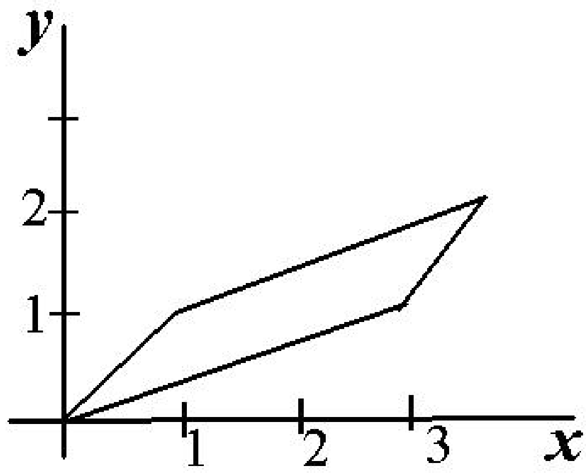
$$\text{Area}(S) = \iint_S 1\, dA = \iint_D 1\, abr\, d\theta\, dr$$



$$\begin{aligned}\text{Area}(S) &= \iint_S 1\,dA = \iint_D 1\,abr\,d\theta\,dr \\ &= \int_0^1 \int_0^{2\pi} abr\,d\theta\,dr \\ &= \pi ab\end{aligned}$$

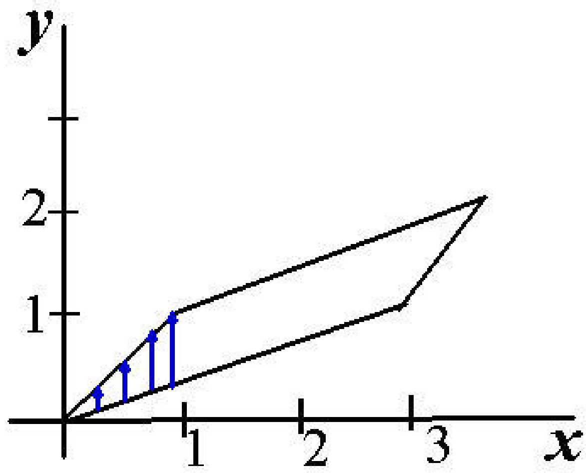
Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\text{Area}(S) = \iint 1\, dA$$



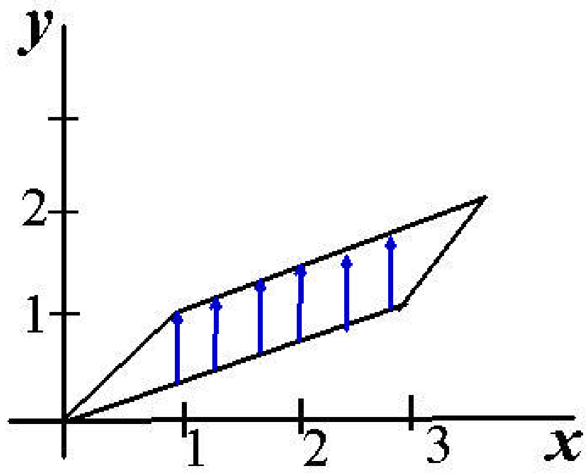
Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\text{Area}(S) = \int\int 1\,dy\,dx$$



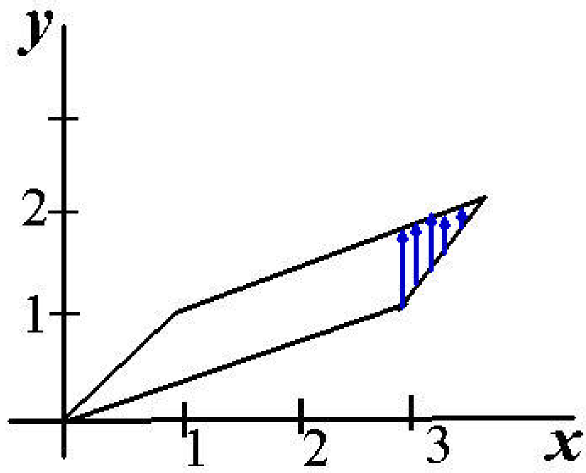
Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\text{Area}(S) = \int\!\!\int 1\, dy\, dx$$



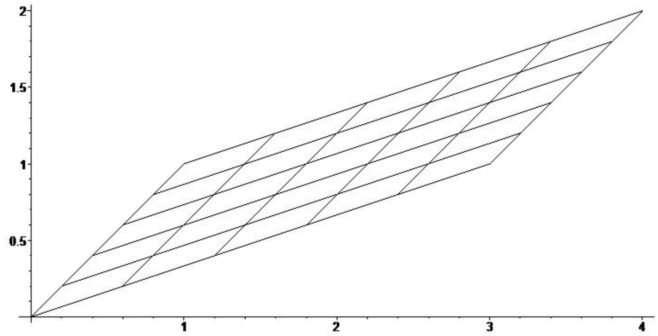
Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\text{Area}(S) = \int\!\!\int 1\,dy\,dx$$



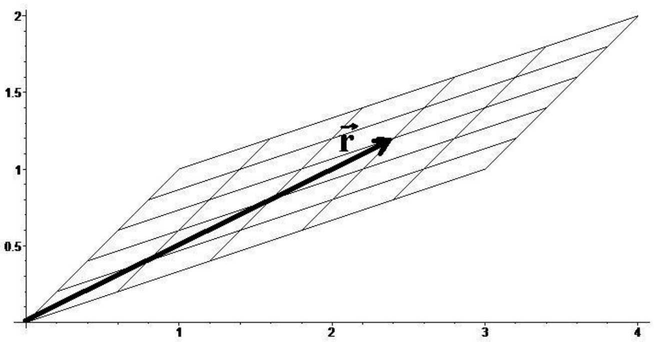
Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\text{Area}(S) = \int\!\!\int 1\, dy\, dx$$



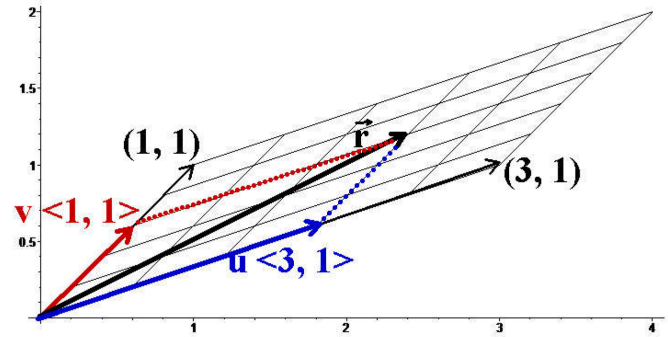
Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\text{Area}(S) = \int\int 1\, dA$$



Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\vec{\mathbf{r}} = u\langle 3,\ 1\rangle + v\langle 1,\ 1\rangle \quad \text{where } 0 \le u,\ v \le 1$$



Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\vec{\mathbf{r}} = u\langle 3,\ 1\rangle + v\langle 1,\ 1\rangle \quad \text{where } 0 \le u,\ v \le 1$$

$$\langle x,\ y\rangle = u\langle 3,\ 1\rangle + v\langle 1,\ 1\rangle = \langle 3u + v,\ u + v\rangle$$

Let S be the region inside the parallelogram with vertices: $(0,\ 0)$ $(3,\ 1)$ $(1,\ 1)$ $(4,\ 2)$

$$\vec{\mathbf{r}} = u\langle 3,\ 1\rangle + v\langle 1,\ 1\rangle \quad \text{where } 0 \le u,\ v \le 1$$

$$\langle x,\ y\rangle = u\langle 3,\ 1\rangle + v\langle 1,\ 1\rangle = \langle 3u + v,\ u + v\rangle$$

$$x = 3u + v \qquad y = u + v$$

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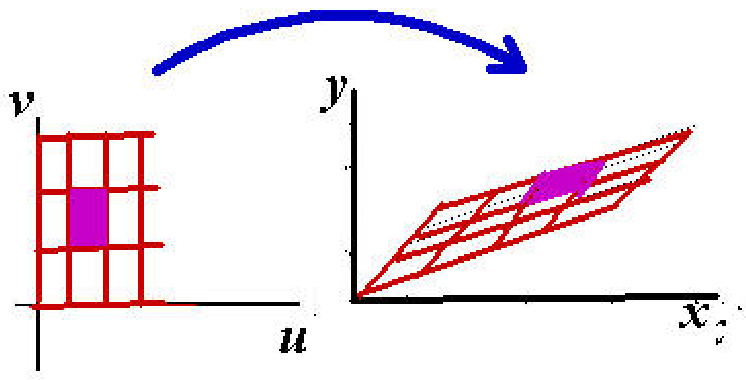
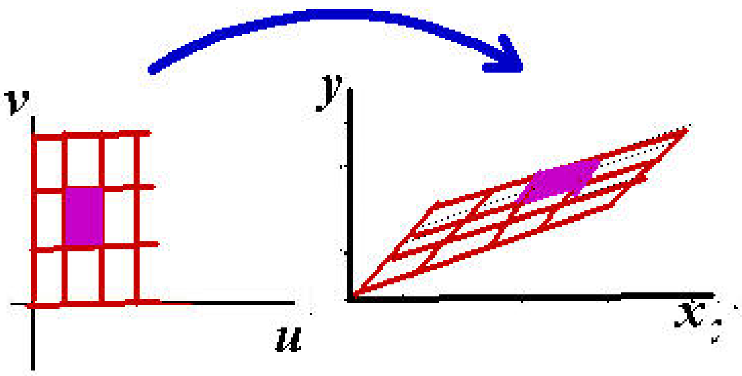


$$x = 3u + v \qquad y = u + v$$

$$dA = J\,du\,dv = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$



$$x = 3u + v \qquad y = u + v$$

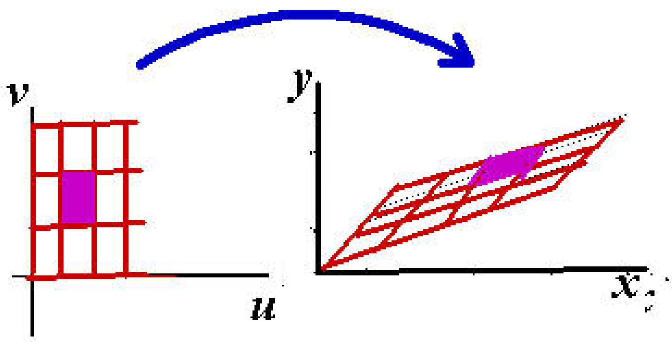
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = 3u + v \qquad y = u + v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ \\ 1 & 1 \end{vmatrix} = 2$$

$$x = 3u + v \qquad y = u + v$$

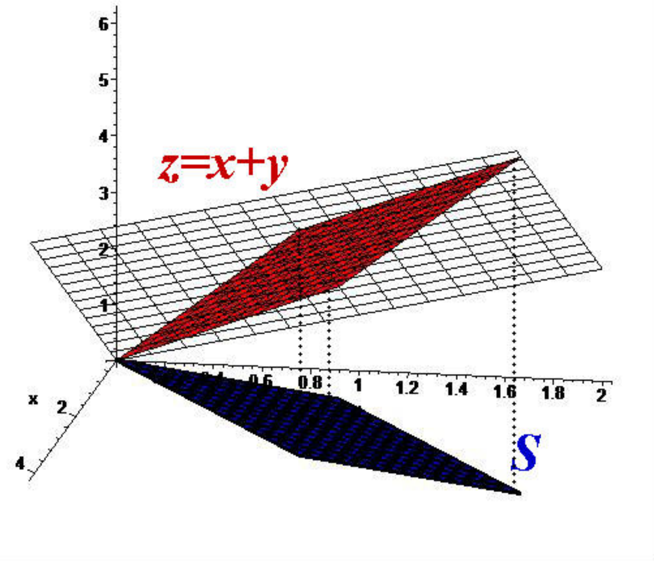
$$dA = \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = 2\, du\, dv$$



$$x = 3u + v \qquad y = u + v$$

$$\text{Area}(S) = \iint_S 1\, dA = \int_0^1 \int_0^1 2\, du\, dv = 2$$

$$\iint_S (x+y)\,dA$$



;

$$z = x + y = (3u + v) + (u + v) = 4u + 2v$$

$$dA = J\,du\,dv = 2\,du\,dv$$

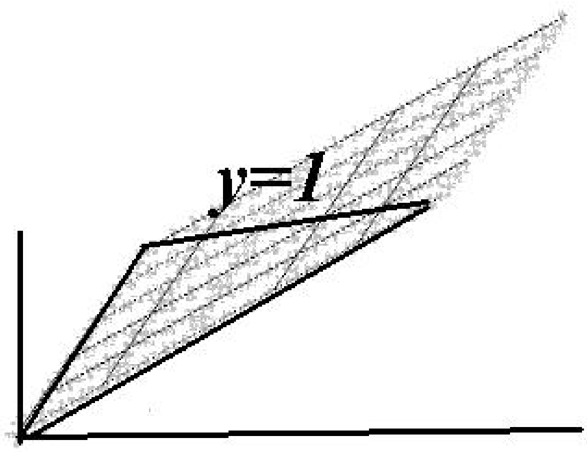
$$\iint_S (x + y)\,dA = \int_0^1 \int_0^1 (4u + 2)\,2\,du\,dv$$

$$z = x + y = (3u + v) + (u + v) = 4u + 2v$$

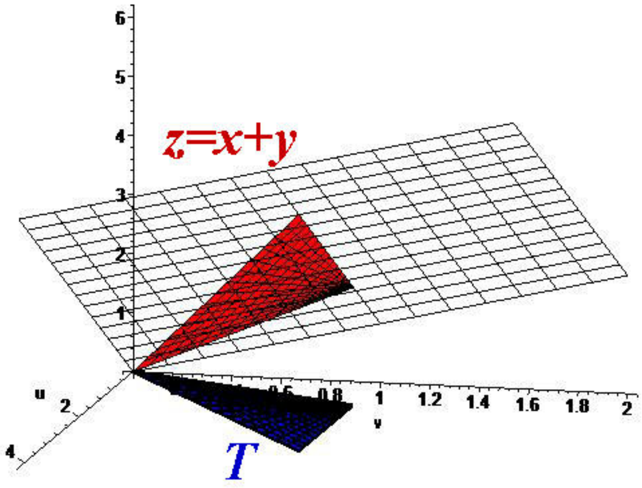
$$dA = J\,du\,dv = 2\,du\,dv$$

$$\begin{aligned}
\iint_S (x+y)\,dA &= \int_0^1 \int_0^1 (4u + 2v)\,2\,du\,dv \\
&= \int_0^1 \left[2u^2 + 2uv\right]_{u=0}^{1} 2dv \\
&= \int_0^1 (4 + 4v)\,dv \\
&= 6
\end{aligned}$$

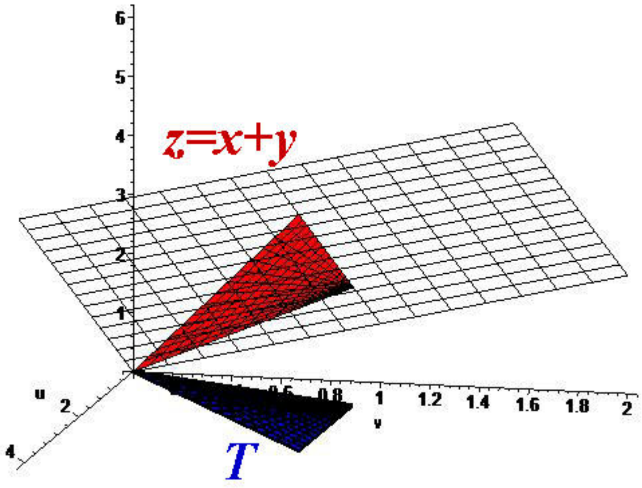
Let T be the region inside S and below $y = 1$



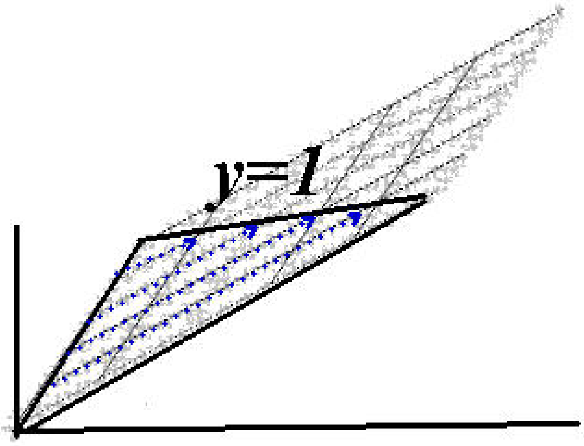
$$\iint_T (x+y)\, dA$$



$$\iint_T (x+y)\,dA = \int\int_D (4u+2v)\,2\,du\,dv$$

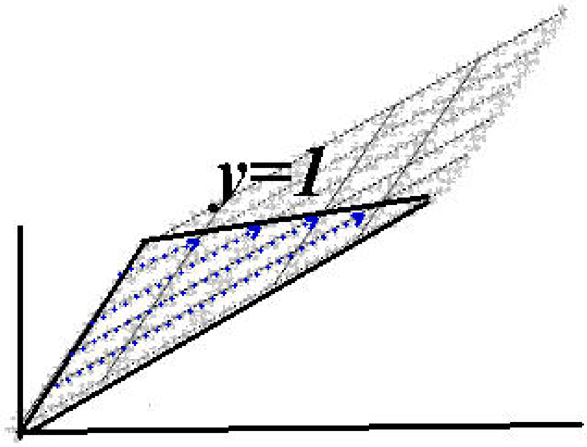


$$\iint_T (x+y)\,dA = \iint_D (4u+2v)\,2\,du\,dv$$

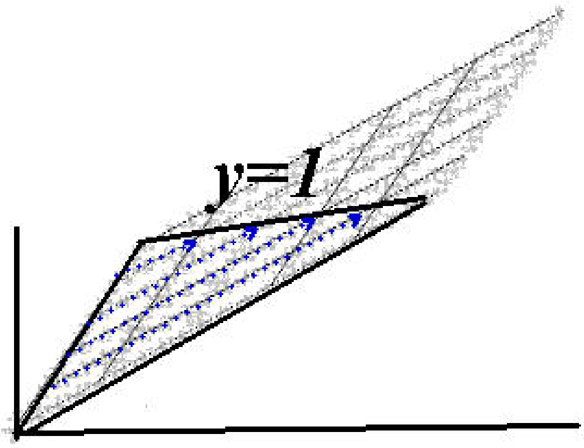


$$x = 3u + v \qquad y = u + v$$

The upper u limit occurs on the line $y = 1$ which imples that $u + v = 1$ and therefore, $u = 1 - v$.



$$\iint_T (x+y)\,dA = \int_0^1 \int_0^{1-v} (4u+2v)\,2\,du\,dv$$



$$\iint_T (x + y)\, dA = \int_0^1 \int_0^{1-v} (4u + 2v)\, 2\, du\, dv = 2$$

Change of Variable Theorem:

If $x = x(u, v)$ and $y = y(u, v)$ then the integral:

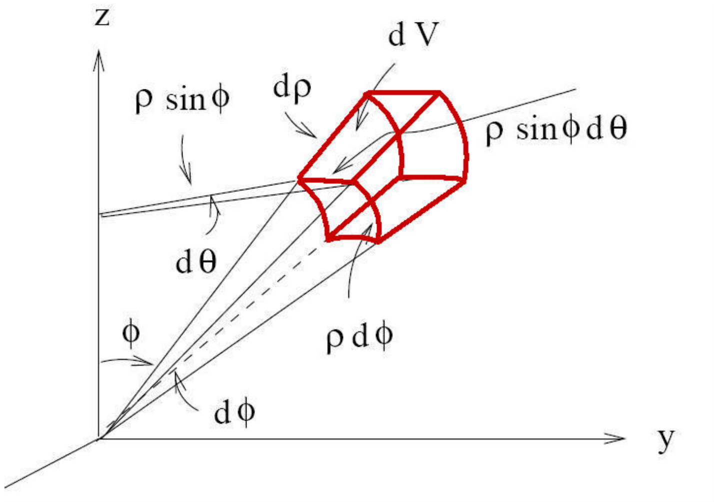
$$\iint_S f(x, y)\, dA$$

can be converted to uv coordinates:

$$\iint_D f(x(u, v), y(u, v))\, \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv$$

$$x = \rho \cos\theta \sin\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\phi$$

$$dV = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}\, d\rho\, d\phi\, d\theta$$



$$x = \rho \cos\theta \sin\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\phi$$

$$dV = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \, d\rho \, d\phi \, d\theta$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} x_\rho & y_\rho & z_\rho \\ x_\phi & y_\phi & z_\phi \\ x_\theta & y_\theta & z_\theta \end{vmatrix}$$

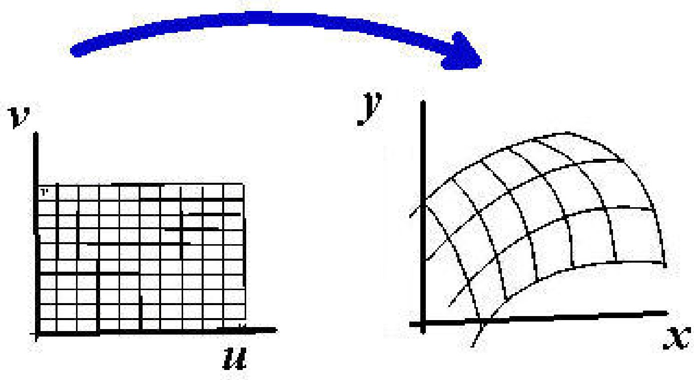
$$x = \rho \cos\theta \sin\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\phi$$

$$dV = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}\, d\rho\, d\phi\, d\theta$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} x_\rho & y_\rho & z_\rho \\ x_\phi & y_\phi & z_\phi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = \rho^2 \sin\phi$$

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$x = x(u, v) \qquad y = y(u, v)$$



$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$