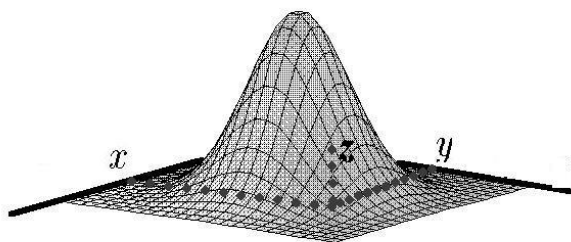


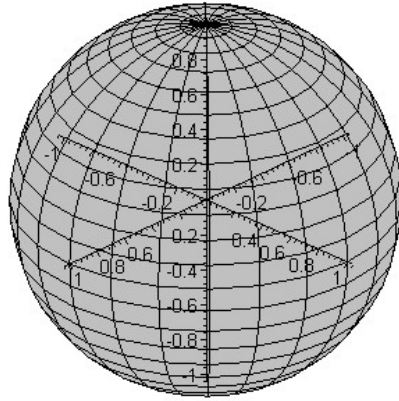
Surface Area - Parametric

$$\text{Area}(S) = \iint_{\mathcal{D}} \sqrt{z_x^2 + z_y^2 + 1} \, dx \, dy$$



$$x^2 + y^2 + z^2 = 1$$

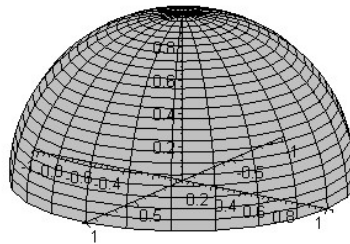
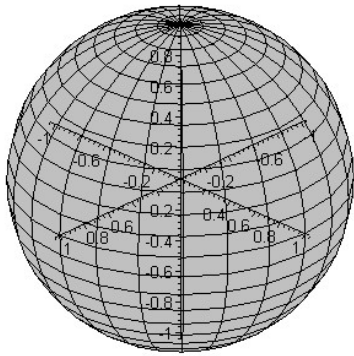
This can't be written in the form $z = f(x, y)$



$$x^2 + y^2 + z^2 = 1$$

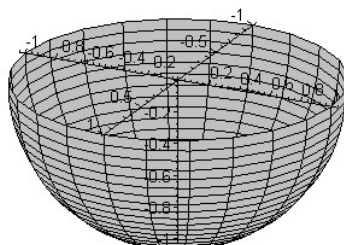
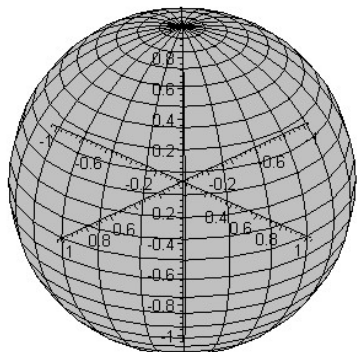
This can't be written in the form $z = f(x, y)$

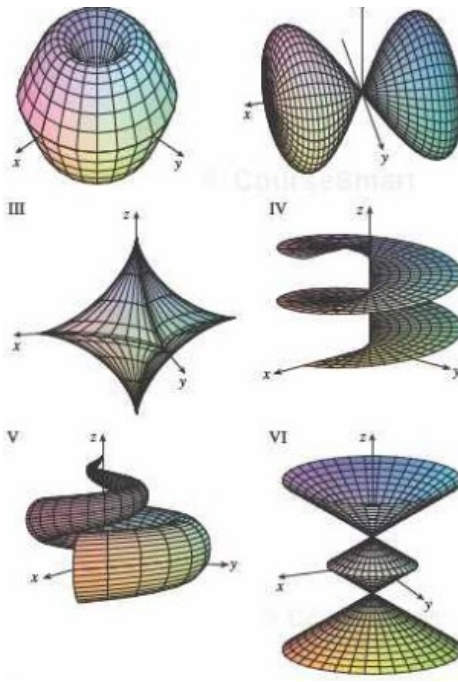
$$z = \sqrt{1 - x^2 - y^2}$$



This can't be written in the form $z = f(x, y)$

$$z = -\sqrt{1 - x^2 - y^2}$$

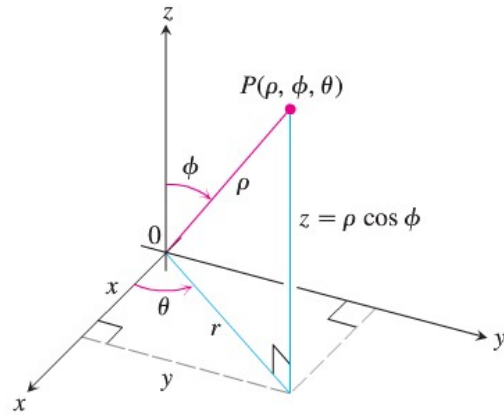




$$x = \rho \cos \theta \sin \phi$$

$$y = \rho \sin \theta \sin \phi$$

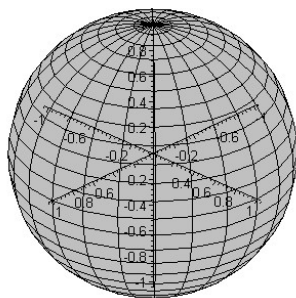
$$z = \rho \cos \phi$$



$$x = a \cos \theta \sin \phi$$

$$y = a \sin \theta \sin \phi$$

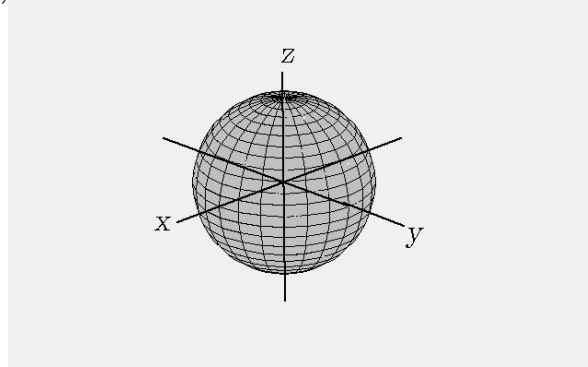
$$z = a \cos \phi$$



$$x = a \cos \theta \sin \phi \quad y = a \sin \theta \sin \phi \quad z = a \cos \phi$$

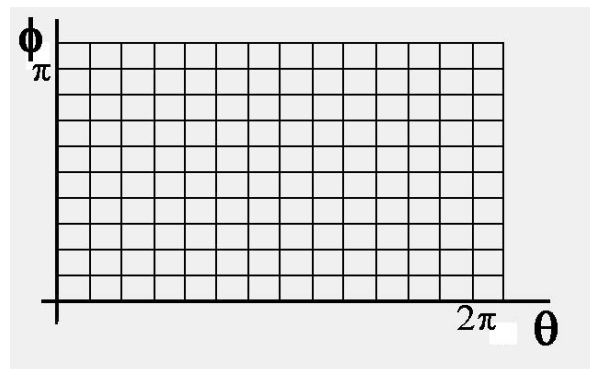
In vector form,

$$\vec{\mathbf{r}} = \langle x, \ y, \ z \rangle = \langle a \cos \theta \sin \phi, \ a \sin \theta \sin \phi, \ a \cos \phi \rangle$$

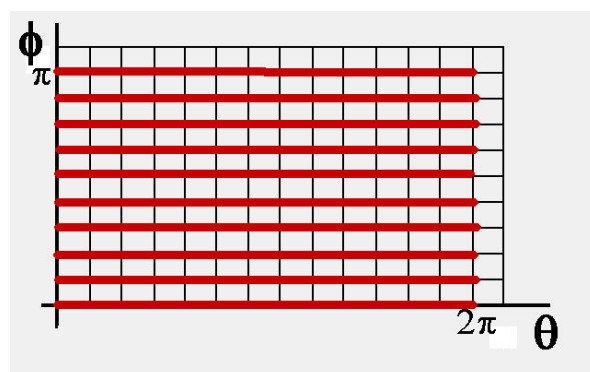


$$0 \leq \theta \leq 2\pi$$

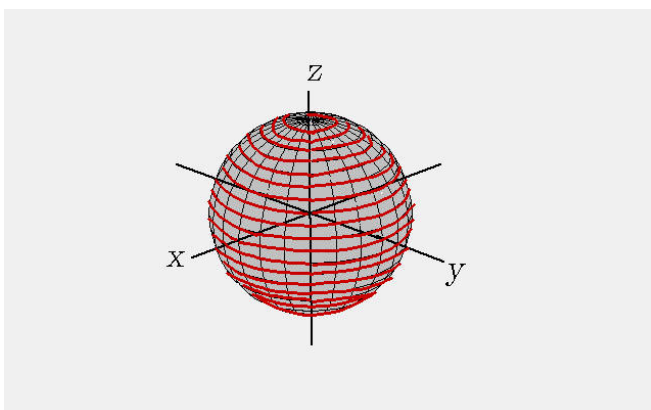
$$0 \leq \phi \leq \pi$$



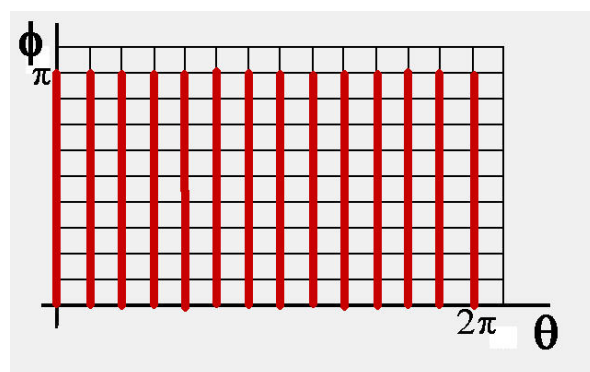
Hold ϕ constant and vary θ



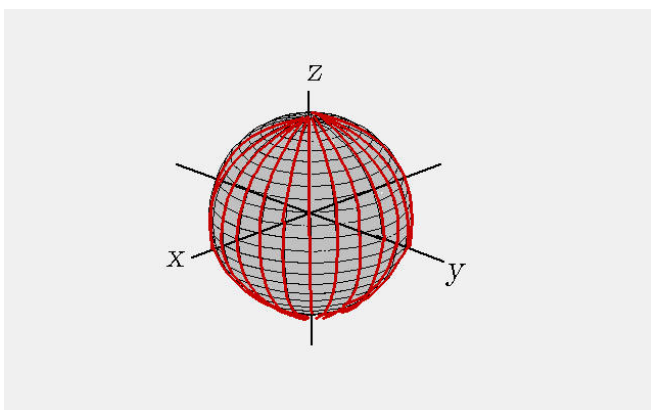
Hold ϕ constant and vary θ



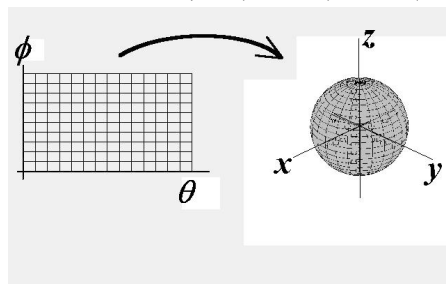
Hold θ constant and vary ϕ



Hold θ constant and vary ϕ



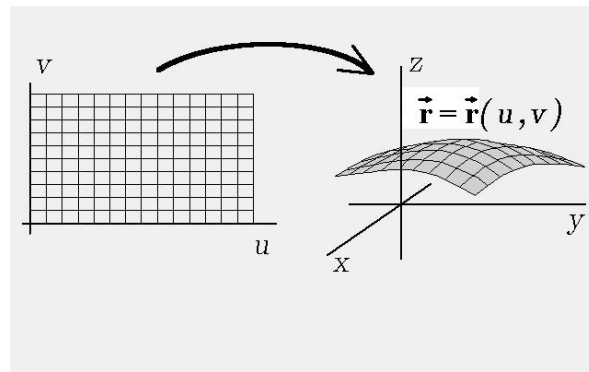
$$\vec{\mathbf{r}} = \langle x(\phi, \theta), y(\phi, \theta), z(\phi, \theta) \rangle$$



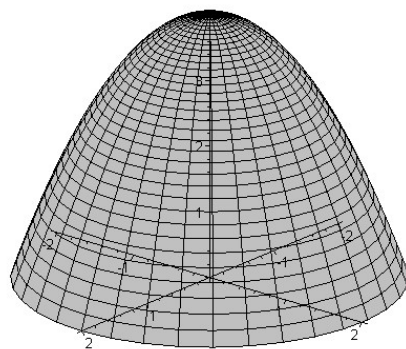
More generally, surfaces can be described by a vector equation of the form

$$\vec{\mathbf{r}} = \langle x(u, v), y(u, v), z(u, v) \rangle$$

The variables u and v are called the parameters



$$z = 4 - x^2 - y^2$$



Polar Coordinates

$$x = r \cos \theta \qquad y = r \sin \theta$$

Polar Coordinates

$$x = u \cos v \qquad y = u \sin v$$

$$z = 4 - x^2 - y^2 = 4 - (x^2 + y^2)$$

$$\text{Let } x = u \cos v \quad y = u \sin v$$

$$z = 4 - ((u \cos v)^2 + (u \sin v)^2) = 4 - u^2$$

$$z = 4 - x^2 - y^2 = 4 - (x^2 + y^2)$$

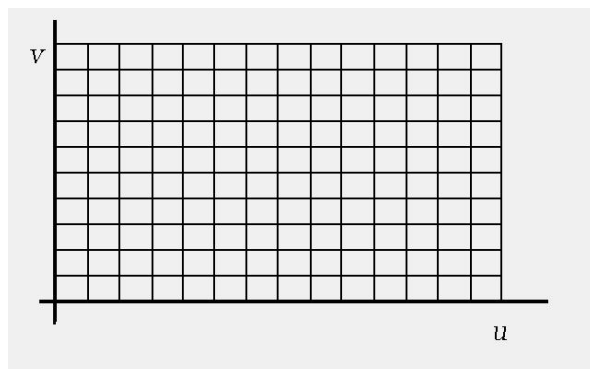
$$\text{Let } x = u \cos v \quad y = u \sin v$$

$$z = 4 - ((u \cos v)^2 + (u \sin v)^2) = 4 - u^2$$

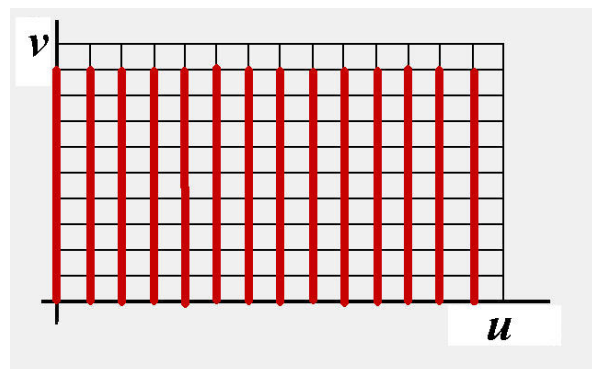
$$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle u \cos v, u \sin v, 4 - u^2 \rangle$$

$$\text{where } 0 \leq v \leq 2\pi \text{ and } 0 \leq u \leq 2$$

$$0 \leq v \leq 2\pi \text{ and } 0 \leq u \leq 2$$



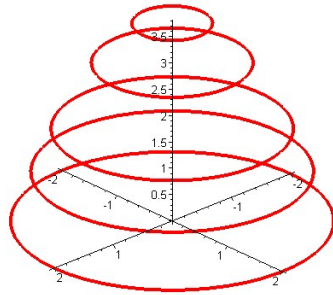
$$0 \leq v \leq 2\pi \text{ and } 0 \leq u \leq 2$$



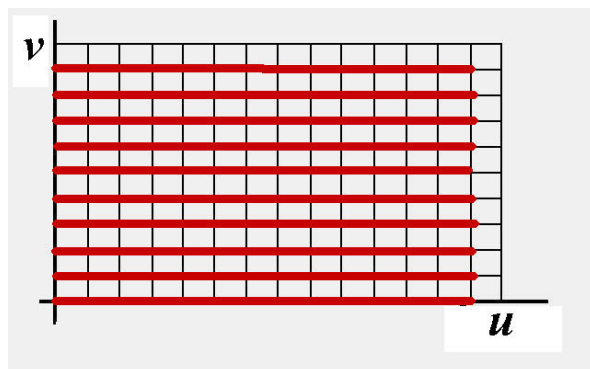
$$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle u \cos v, u \sin v, 4 - u^2 \rangle$$

where $0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$

Vary v for different values of u



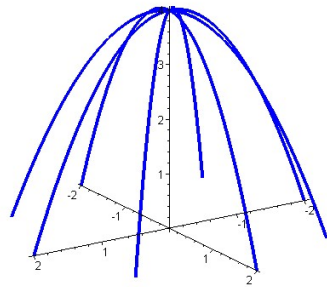
$$0 \leq v \leq 2\pi \text{ and } 0 \leq u \leq 2$$



$$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle u \cos v, u \sin v, 4 - u^2 \rangle$$

where $0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$

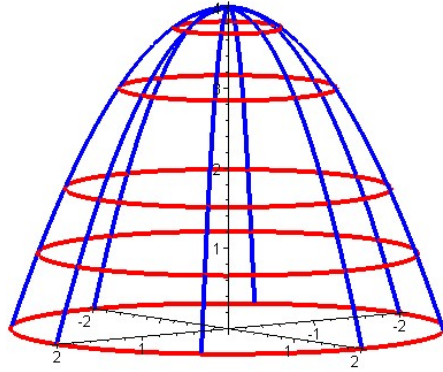
Vary u for different values of v



$$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle u \cos v, u \sin v, 4 - u^2 \rangle$$

where $0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$

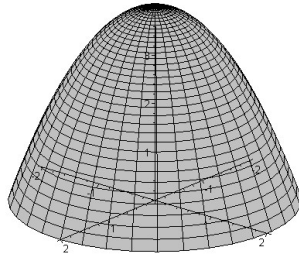
Combine the grid lines together



$$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle u \cos v, u \sin v, 4 - u^2 \rangle$$

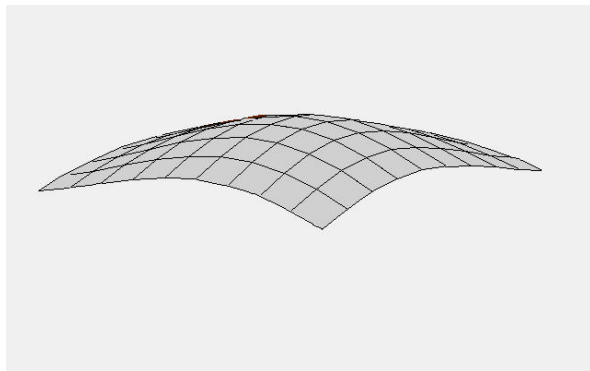
where $0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$

We have induced a grid on the parabolic surface

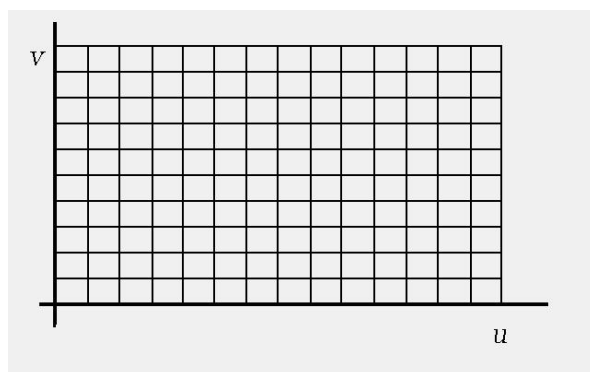


Area of the Surface Element

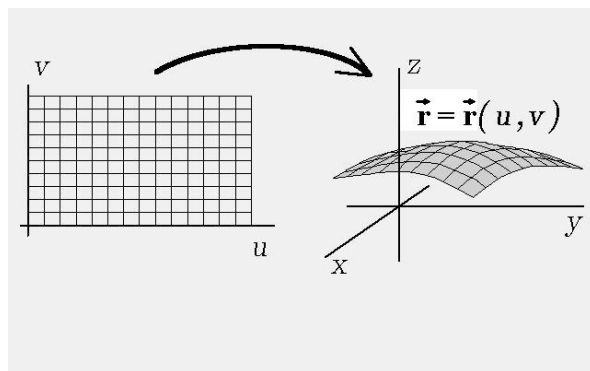
The equation $\vec{\mathbf{r}}(u, v) = \langle x, y, z \rangle$ represents a transformation from a uv -plane to an xyz -axis. The graph of all the points $\vec{\mathbf{r}}(u, v)$ is a surface.



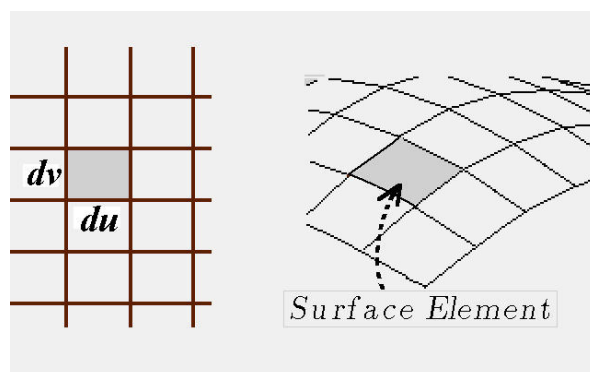
Suppose we subdivide a region of the uv -axis into a rectangular grid.



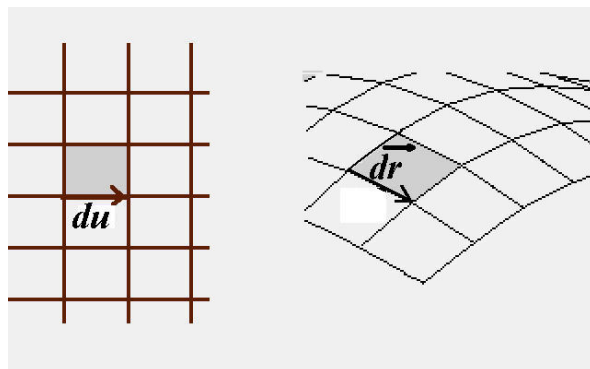
This will automatically subdivide the surface $\vec{\mathbf{r}}(u, v)$ into a grid.



Consider one rectangle on the uv grid.

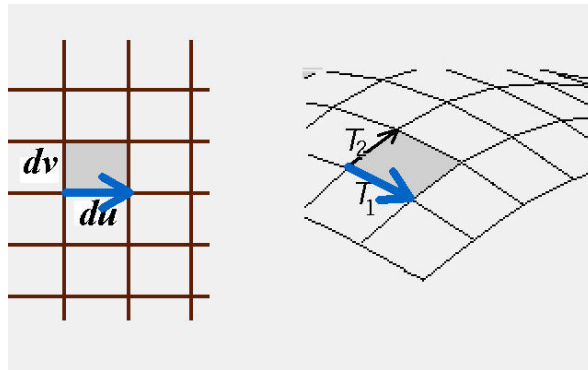


Let $d\vec{r}$ be the vector denoting the change in position along this curve.

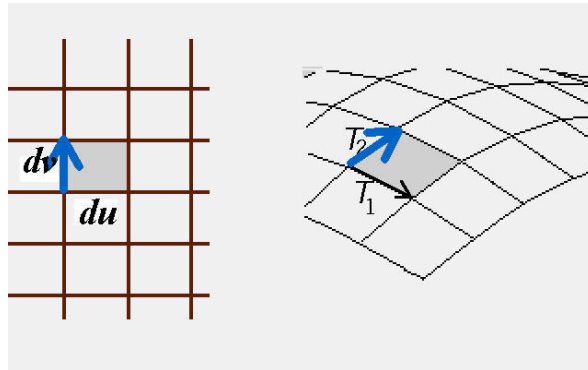


$$\text{Let } \vec{\mathbf{T}}_1 = \mathbf{d}\vec{\mathbf{r}} = \frac{\partial \vec{\mathbf{r}}}{\partial u} du$$

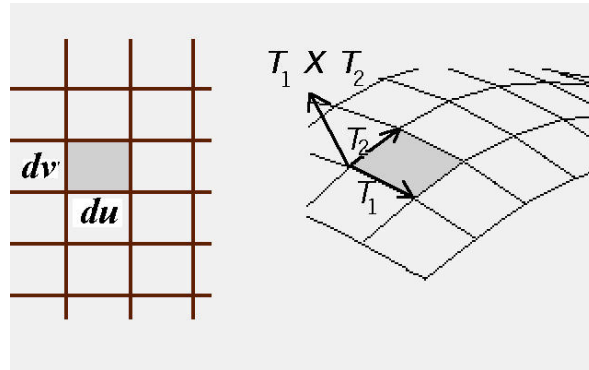
$$\vec{\mathbf{T}}_1 = \frac{\partial \vec{\mathbf{r}}}{\partial u} du = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle du$$



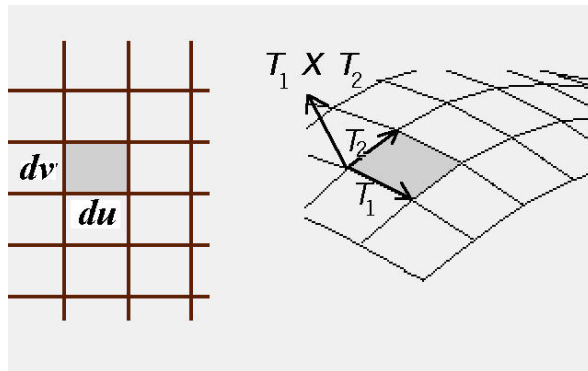
$$\vec{\mathbf{T}}_2 = \frac{\partial \vec{\mathbf{r}}}{\partial v} dv = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle dv$$



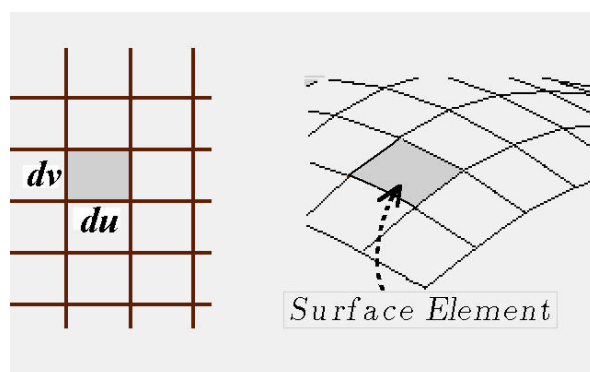
$$\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2 = \left(\frac{\partial \vec{\mathbf{r}}}{\partial u} du \right) \times \left(\frac{\partial \vec{\mathbf{r}}}{\partial v} dv \right)$$



$$\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2 = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du \, dv$$

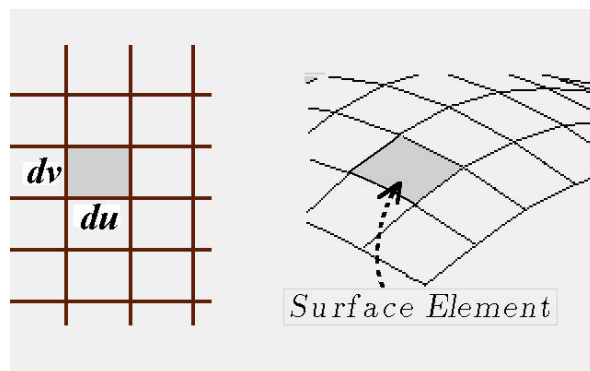


The length of this cross product is equal to the area of one surface element

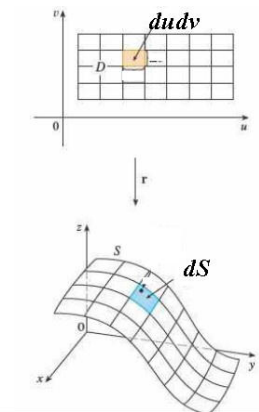


The length of this cross product is denoted by dS

$$dS = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv$$

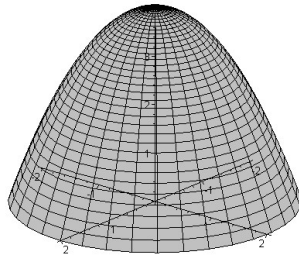


Total Surface Area of $S = \iint_{\mathcal{D}} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv$



$$\vec{\mathbf{r}} = \langle u \cos v, \ u \sin v, \ 4 - u^2 \rangle$$

where $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$



$$\vec{\mathbf{r}} = \langle u \cos v, \ u \sin v, \ 4 - u^2 \rangle$$

$$\vec{\mathbf{r}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u} = \langle \cos v, \ \sin v, \ -2u \rangle$$

$$\vec{\mathbf{r}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v} = \langle -u \sin v, \ u \cos v, \ 0 \rangle$$

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \cos v & \sin v & -2u \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$\vec{\mathbf{r}} = \langle u \cos v, \ u \sin v, \ 4 - u^2 \rangle$$

$$\vec{\mathbf{r}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u} = \langle \cos v, \ \sin v, \ -2u \rangle$$

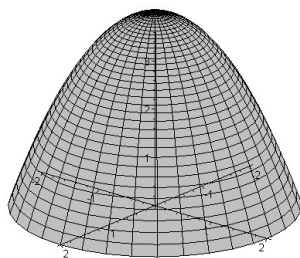
$$\vec{\mathbf{r}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v} = \langle -u \sin v, \ u \cos v, \ 0 \rangle$$

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2u^2 \cos v, \ 2u^2 \sin v, \ u \rangle$$

$$|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| = \sqrt{4u^4 + u^2} = u\sqrt{4u^2 + 1}$$

Let S be the portion of the paraboloid for $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$ Find the surface area

$$A(S) = \iint_{\mathcal{D}} |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dv \, du$$



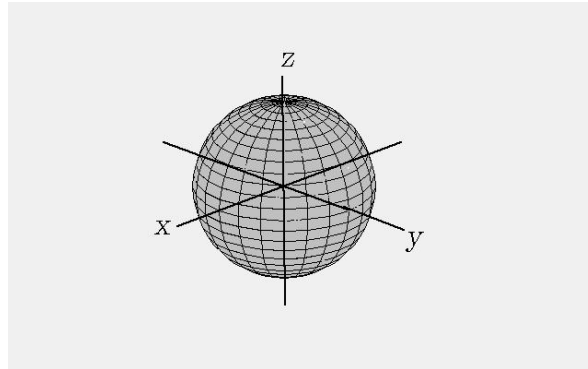
$$\begin{aligned}
A(S) &= \iint_{\mathcal{D}} |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dv \, du \\
&= \int_0^2 \int_0^{2\pi} u \sqrt{4u^2 + 1} \, dv \, du \\
&= \int_0^2 2\pi u \sqrt{4u^2 + 1} \, du \\
&= \frac{\pi}{6} (17\sqrt{17} - 1)
\end{aligned}$$

Find the surface area of a sphere of radius a

$$x = a \cos \theta \sin \phi \quad y = a \sin \theta \sin \phi \quad z = a \cos \phi$$

In vector form,

$$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle a \cos \theta \sin \phi, a \sin \theta \sin \phi, a \cos \phi \rangle$$



$$A(S) = \int \int_{\mathcal{D}} |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dv \, du$$

$$A(S) = \int_0^\pi \int_0^{2\pi} |\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta| \, d\theta \, d\phi$$

$$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle a \cos \theta \sin \phi, a \sin \theta \sin \phi, a \cos \phi \rangle$$

$$\vec{\mathbf{r}}_\phi = \langle a \cos \theta \cos \phi, a \sin \theta \cos \phi, -a \sin \phi \rangle$$

$$\vec{\mathbf{r}}_\theta = \langle -a \sin \theta \sin \phi, a \cos \theta \sin \phi, 0 \rangle$$

Calculate $\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta$

$$\vec{\mathbf{r}}_\phi = \langle a \cos \theta \cos \phi, \ a \sin \theta \cos \phi, \ -a \sin \phi \rangle$$

$$\vec{\mathbf{r}}_\theta = \langle -a \sin \theta \sin \phi, \ a \cos \theta \sin \phi, \ 0 \rangle$$

$$\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a \cos \theta \cos \phi & a \sin \theta \cos \phi & -a \sin \phi \\ -a \sin \theta \sin \phi & a \cos \theta \sin \phi & 0 \end{vmatrix}$$

$$\vec{\mathbf{r}}_\phi = \langle a \cos \theta \cos \phi, \ a \sin \theta \cos \phi, \ -a \sin \phi \rangle$$

$$\vec{\mathbf{r}}_\theta = \langle -a \sin \theta \sin \phi, \ a \cos \theta \sin \phi, \ 0 \rangle$$

$$\begin{aligned} \vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a \cos \theta \cos \phi & a \sin \theta \cos \phi & -a \sin \phi \\ -a \sin \theta \sin \phi & a \cos \theta \sin \phi & 0 \end{vmatrix} \\ &= a^2 \sin \phi \langle \cos \theta \sin \phi, \ \sin \theta \sin \phi, \ \cos \phi \rangle \end{aligned}$$

$$|\langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle| = 1$$

$$\begin{aligned}
 |\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta| &= a^2 \sin \phi |\langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle| \\
 &= a^2 \sin \phi
 \end{aligned}$$

$$dS = |\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta| \, d\theta \, d\phi = a^2 \sin \phi \, d\theta \, d\phi$$

$$\begin{aligned} A(S) &= \int_0^\pi \int_0^{2\pi} a^2 \sin \phi \, d\theta \, d\phi \\ &= 2\pi a^2 \int_0^\pi \sin \phi \, d\phi \\ &= 4\pi a^2 \end{aligned}$$

