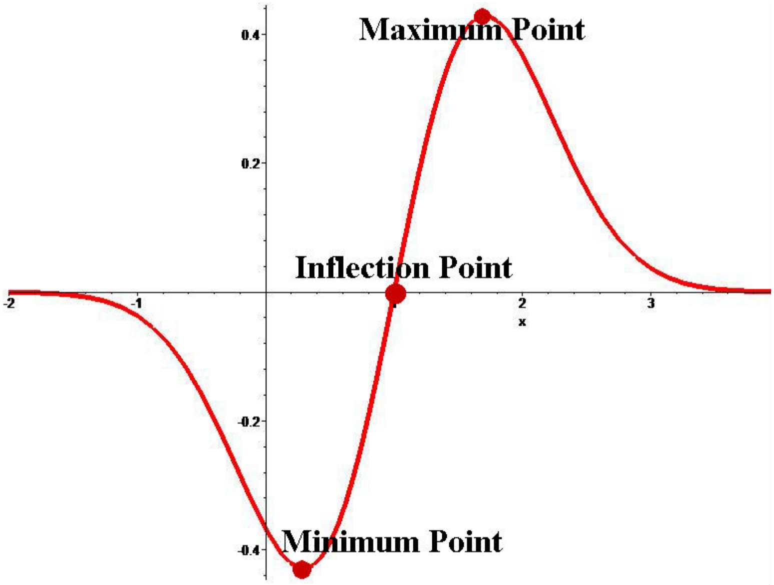
Critical Points for $z = f(x, y)$



$$f(x) = xe^{-x}$$

Find all maximums, minimums, points of inflection and use this information to sketch the curve.

$$f(x) = xe^{-x}$$

$$f'(x) = x\left(-e^{-x}\right) + (1)e^{-x} = (1 - x)e^{-x}$$

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$$f'(x) = 0 \quad \text{when} \quad x = 1$$

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$$f''(x) = (1 - x)\left(-e^{-x}\right) + (-1)e^{-x} = (x - 2)e^{-x}$$

$$f(x) = xe^{-x}$$

$$f'(x) = (1 - x)e^{-x}$$

$$f'(x) = 0 \quad \text{when} \quad x = 1$$

$$f''(x) = (x - 2)e^{-x}$$

$$f''(1) = (1 - 2)e^{-1} = -\frac{1}{e}$$

Therefore, $\left(1,\ \frac{1}{e}\right)$ is a maximum point.

$$f(x) = xe^{-x}$$

$$f'(x) = (1 - x)e^{-x}$$

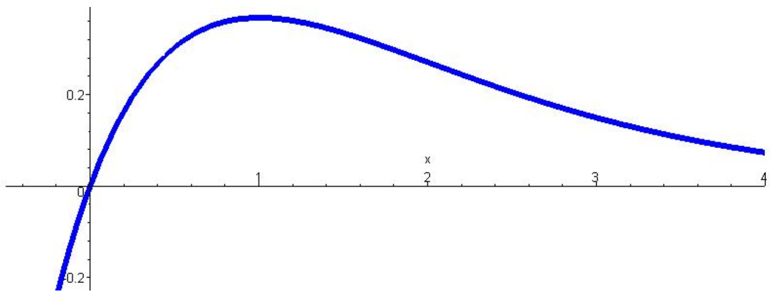
$$f''(x) = (x - 2)e^{-x}$$

$$f''(x) = 0 \quad \text{when} \quad x = 2$$

Therefore, $\left(2, \frac{2}{e^2}\right)$ is an inflection point.

$\left(1,\ \frac{1}{e}\right)$ is a maximum point. $\left(2, \frac{2}{e^2}\right)$ is an inflection point.

$$f(x) = x\,e^{-x}$$



Find the critical points of $z = 4x - x^2 - 2y^2$ and sketch the surface.

$$z = 4x - x^2 - 2y^2$$

$$\frac{\partial z}{\partial x} = 4 - 2x \qquad \frac{\partial z}{\partial y} = -4y$$

$$\frac{\partial z}{\partial x} = 0 \quad \text{when} \quad x = 2$$

$$\frac{\partial z}{\partial y} = 0 \quad \text{when} \quad y = 0$$

Therefore, there is a critical point at $(x,\ y) = (2,\ 0)$.

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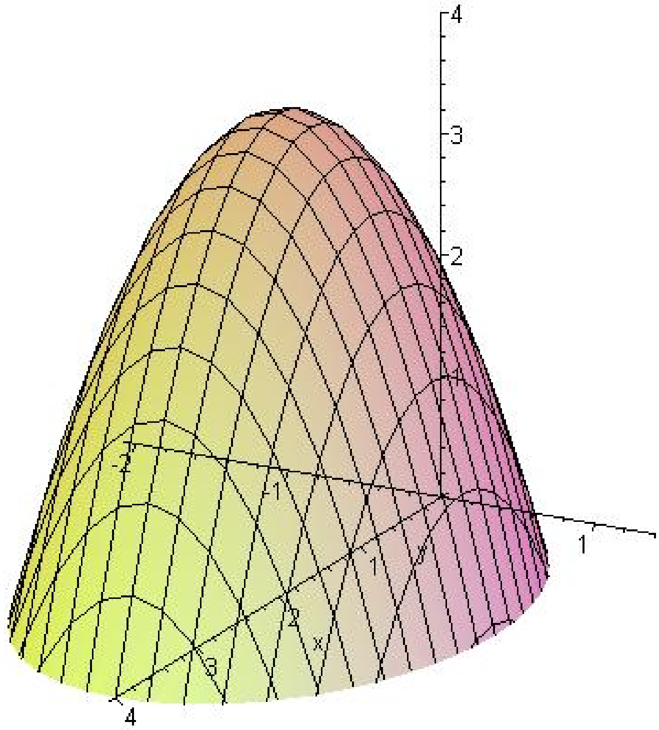
$$\frac{\partial^2 z}{\partial x^2} = -2 \qquad\qquad \frac{\partial^2 z}{\partial y^2} = -4$$



$$z = 4x - x^2 - 2y^2$$

$$z = x^2 - 2y^2 - 2x + 6$$

Find the maximums and minimums

$$\frac{\partial z}{\partial x} = 2x - 2 \qquad \frac{\partial z}{\partial y} = -4y$$

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The point $(1,\ 0)$ is the only critical point. Will this give us a maximum or a minimum?

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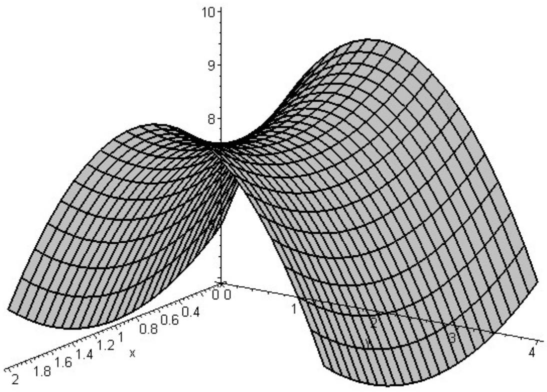
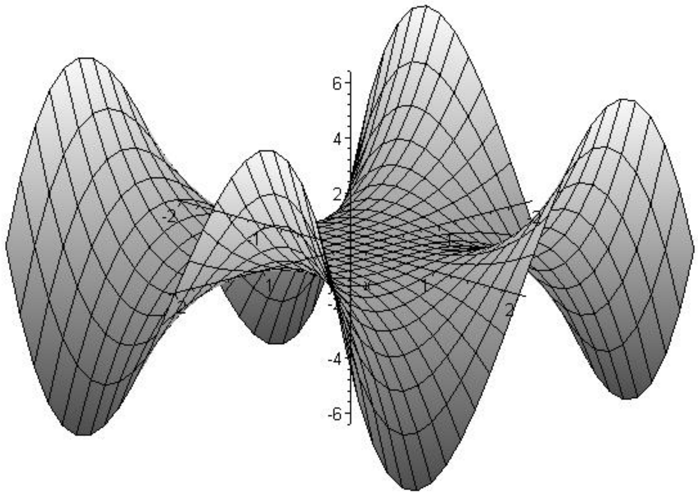
The point $(1,\ 0)$ is the only critical point. Will this give us a maximum or a minimum?

$$\frac{\partial^2 z}{\partial x^2} = 2 \qquad \frac{\partial^2 z}{\partial y^2} = -4$$

$$z = x^2 - 2y^2 - 2x + 6$$

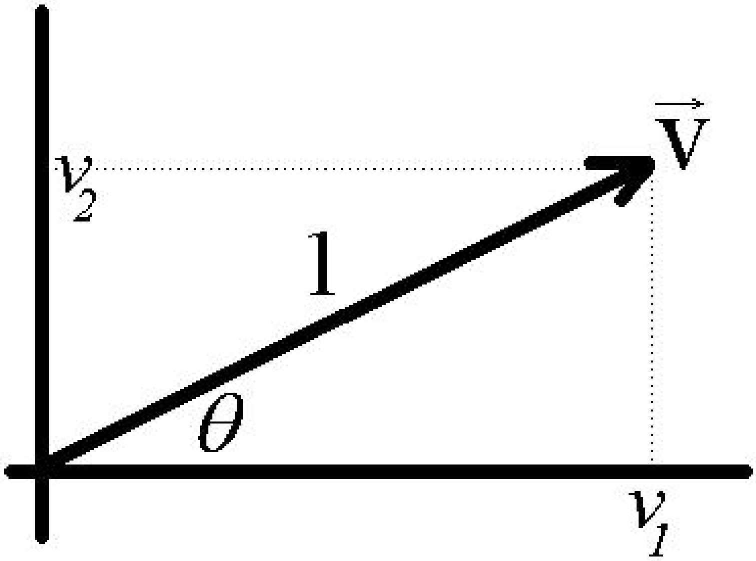
The point $(1,\ 0, 5)$ is a *saddle point*





$$z = x^3y - y^3x$$

Let $\vec{\mathbf{v}}$ be a unit vector making an angle θ with the positive x-axis.



$$\vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle = \langle \cos\theta,\ \sin\theta \rangle$$

The directional derivative can be written in terms of θ

$$\begin{aligned} D_{\vec{\mathbf{v}}} f &= \nabla f \bullet \vec{\mathbf{v}} \\ &= \frac{\partial f}{\partial x} \cdot v_1 + \frac{\partial f}{\partial y} \cdot v_2 \\ &= \frac{\partial f}{\partial x} \cos\theta + \frac{\partial f}{\partial y} \sin\theta \end{aligned}$$

$$D_{\vec{\mathbf{v}}} f = \frac{\partial f}{\partial x} \cos\theta + \frac{\partial f}{\partial y} \sin\theta$$

$$D_{\vec{\mathbf{v}}} \left(\text{function}\right) = \frac{\partial}{\partial x} \left(\text{function}\right) \cos\theta + \frac{\partial}{\partial y} \left(\text{function}\right) \sin\theta$$

$$D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}}f\right) = \frac{\partial}{\partial x}\left(D_{\vec{\mathbf{v}}}f\right)\cos\theta + \frac{\partial}{\partial y}\left(D_{\vec{\mathbf{v}}}f\right)\sin\theta$$

This is the same as:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)\sin\theta$$

$$D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}}f\right)=\frac{\partial}{\partial x}\left(D_{\vec{\mathbf{v}}}f\right)\cos\theta+\frac{\partial}{\partial y}\left(D_{\vec{\mathbf{v}}}f\right)\sin\theta$$

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$$D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}}f\right)=\frac{\partial^2 f}{\partial x^2}\cos^2\theta+2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin^2\theta$$

$$D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}} f\right) = f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta$$

$$
\begin{aligned}
D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}} f\right) &= f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta \\
&= f_{xx}\sin^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta} + 2\frac{f_{xy}}{f_{xx}}\frac{\cos\theta}{\sin\theta} + \frac{f_{yy}}{f_{xx}}\right) \\
&= f_{xx}\sin^2\theta\left(\cot^2\theta + 2\frac{f_{xy}}{f_{xx}}\cot\theta + \frac{f_{yy}}{f_{xx}}\right)
\end{aligned}
$$

Complete the square:

$$\cot^2\theta + 2\frac{f_{xy}}{f_{xx}}\cot\theta + \frac{f_{yy}}{f_{xx}}$$

This is the same as:

$$\cot^2\theta + 2\frac{f_{xy}}{f_{xx}}\cot\theta + \frac{f_{xy}^2}{f_{xx}^2} - \frac{f_{xy}^2}{f_{xx}^2} + \frac{f_{yy}}{f_{xx}}$$

Complete the square:

$$\cot^2\theta + 2\frac{f_{xy}}{f_{xx}}\cot\theta + \frac{f_{yy}}{f_{xx}}$$

This is the same as:

$$\left(\cot\theta + \frac{f_{xy}}{f_{xx}}\right)^2 - \frac{f_{xy}^2}{f_{xx}^2} + \frac{f_{yy}}{f_{xx}}$$

Complete the square:

$$\cot^2\theta + 2\frac{f_{xy}}{f_{xx}}\cot\theta + \frac{f_{yy}}{f_{xx}}$$

This is the same as:

$$\left(\cot\theta + \frac{f_{xy}}{f_{xx}}\right)^2 + \frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}$$

$$D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}}f\right) = f_{xx}\sin^2\theta\left(\cot^2\theta + 2\frac{f_{xy}}{f_{xx}}\cot\theta + \frac{f_{yy}}{f_{xx}}\right)$$

$$= f_{xx}\sin^2\theta\left(\left(\cot\theta + \frac{f_{xy}}{f_{xx}}\right)^2 + \frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}\right)$$

$$f_{xx}\boxed{\sin^2\theta}\left(\boxed{\left(\cot\theta+\frac{f_{xy}}{f_{xx}}\right)^2}+\frac{f_{xx}f_{yy}-f_{xy}^2}{\boxed{f_{xx}^2}}\right)$$

non-negative

could be negative

$$\boxed{f_{xx}}\,\boxed{\sin^2\theta}\left(\boxed{\left(\cot\theta+\frac{f_{xy}}{f_{xx}}\right)^2}+\frac{\boxed{f_{xx}f_{yy}-f_{xy}^2}}{\boxed{f_{xx}^2}}\right)$$

non-negative

The Hessian determinant

$$\mathcal{H} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

What if $f_{xx}f_{yy} - f_{xy}^2$ is negative?

$$f_{xx}\sin^2\theta\left(\boxed{\left(\cot\theta+\frac{f_{xy}}{f_{xx}}\right)^2}+\boxed{\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{xx}^2}}\right)$$

Could be any number from 0 to infinity

negative if numerator is negative

What if $f_{xx} f_{yy} - f_{xy}^2$ is negative?

Then $D_{\vec{\mathbf{v}}} \left(D_{\vec{\mathbf{v}}} f \right)$ will be negative for some values of θ and positive for others.

Saddle point!

What if $f_{xx}f_{yy} - f_{xy}^2$ is positive?

could be negative

$$\boxed{f_{xx}}\,\boxed{\sin^2\theta}\left(\boxed{\left(\cot\theta + \frac{f_{xy}}{f_{xx}}\right)^2} + \boxed{\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}}\right)$$

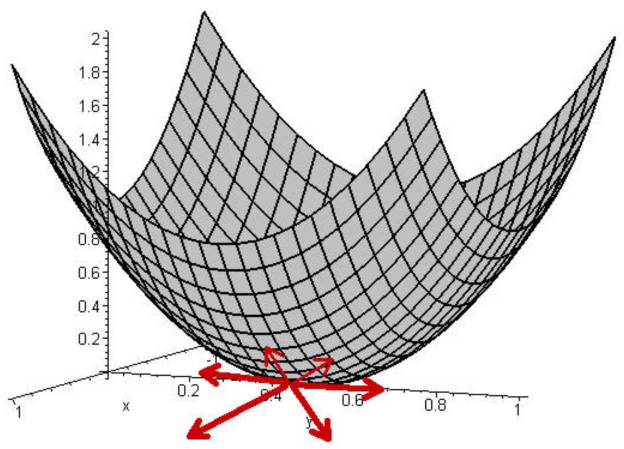
non-negative

What if $f_{xx}f_{yy} - f_{xy}^2$ is positive?

Then $D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}} f\right) > 0$ when f_{xx} is positive
and $D_{\vec{\mathbf{v}}}\left(D_{\vec{\mathbf{v}}} f\right) < 0$ when f_{xx} is negative

$$\mathcal{H} = f_{xx} f_{yy} - f_{xy}^2$$

If $\mathcal{H} > 0$ at a point then the concavity of the surface remains the same in all directions



Theorem:

Suppose $f_x = 0$ and $f_y = 0$ at some point (a, b).

If $\mathcal{H} > 0$ and $f_{xx} > 0$ at (a, b) then $f(x, y)$ has a relative minimum at (a, b)

If $\mathcal{H} > 0$ and $f_{xx} < 0$ at (a, b) then $f(x, y)$ has a relative maximum at (a, b)

If $\mathcal{H} < 0$ at (a, b) then $f(x, y)$ has a saddle point at (a, b)

$$z = 4x - x^2 - 2y^2$$

$$\frac{\partial z}{\partial x} = 4 - 2x \qquad \frac{\partial z}{\partial y} = -4y$$

There is a critical point at $(x,\ y) = (2,\ 0)$.

$$\frac{\partial^2 z}{\partial x^2} = -2 \qquad \frac{\partial^2 z}{\partial y^2} = -4$$

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There is a critical point at $(x,\ y) = (2,\ 0)$.

$$\frac{\partial^2 z}{\partial x^2} = -2 \qquad \frac{\partial^2 z}{\partial y^2} = -4 \qquad \frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\mathcal{H} = f_{xx} f_{yy} - f_{xy}^2 = (-2)(-4) - (0)^2 = 8 > 0$$

At $(2,\ 0)$, the surface reaches a maximum height.

$$z = y^3 + 3y^2 - x^2 + 4x - 4$$

Find and classify the critical points.

$$\frac{\partial z}{\partial x} = -2x + 4 \qquad \frac{\partial z}{\partial y} = 3y^2 + 6y$$

$$z = y^3 + 3y^2 - x^2 + 4x - 4$$

Find and classify the critical points.

$$\frac{\partial z}{\partial x} = -2x + 4 \qquad \frac{\partial z}{\partial y} = 3y^2 + 6y = 3y(y+2)$$

$\frac{\partial z}{\partial x} = 0$ when $x = 2$
$\frac{\partial z}{\partial y} = 3y(y+2) = 0$ at $y = 0$ and at $y = -2$
So, we have critical points at $(2,\ 0)$ and at $(2,\ -2)$

$$z = y^3 + 3y^2 - x^2 + 4x - 4$$

$$\frac{\partial z}{\partial x} = -2x + 4 \qquad \frac{\partial z}{\partial y} = 3y^2 + 6y$$

$$\frac{\partial^2 z}{\partial x^2} = -2 \qquad \frac{\partial^2 z}{\partial y^2} = 6y + 6 \qquad \frac{\partial^2 z}{\partial x \partial y} = 0$$

$$z_{xx} = -2 \qquad z_{yy} = 6y + 6 \qquad z_{xy} = 0$$

At $x = 2$ and $y = 0$, $z_{xx} < 0$ and the Hessian is:

$$\mathcal{H} = z_{xx}z_{yy} - z_{xy}^2 = (-2)(6) - 0^2 = -12$$

Conclusion: There is a saddle point at $(2,\ 0)$

$$z_{xx} = -2 \qquad z_{yy} = 6y + 6 \qquad z_{xy} = 0$$

At $x = 2$ and $y = -2$, $z_{xx} < 0$ and the Hessian is:

$$\mathcal{H} = z_{xx}z_{yy} - z_{xy}^2 = (-2)(-6) - 0^2 = 12$$

Conclusion: There is a maximum point at $(2,\ -2)$

$$z = y^3 + 3y^2 - x^2 + 4x - 4$$