Assignment 14. *Applications to Springs and Circuits*

1. A spring with a spring constant of $k = 16$ N/m (Newtons per meter) is attached to a mass of $m = 2$ kg. Let x be the displacement of the mass relative to the equilibrium position. Suppose the initial displacement is $x(0) = 4$ m and the initial velocity is $v(0) = 0$. Write a differential equation that determines x as a function of t. Solve the differential equation.

$$2\frac{d^2x}{dt^2} + 16x = 0 \qquad x(0) = 4 \qquad x'(0) = 0$$

$$x(t) = 4\cos 2\sqrt{2}t$$

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2. An object of mass 4 kg is attached to a spring with spring constant $k = 100$ N/m. Assume a resistive force due to friction with a damping constant of $b = 32$ kg/sec. If the object begins at the equilibrium position with an initial velocity of 3 m/sec, determine the position of the object as a function of time by solving an appropriate differential equation.

Let $y(t)$ be the displacement relative to the equilibrium position.

$$4\frac{d^2y}{dt^2} + 32\frac{dy}{dt} + 100y = 0 \quad \text{which simplifies to:} \quad \frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 0$$

We can find a solution of the form e^{rt}. If we do this and impose the initial conditions, we obtain:

$$y = e^{-4t}\sin 3t$$

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3. An object of mass 1 kg is attached to a spring with spring constant $k = 4$ N/m. There is a resistive force due to friction with a damping constant of $b = 4$ kg/sec. There is also a time dependent force acting on the object of $F(t) = 3te^{-2t}$. Let $x(t)$ denote the position of the object relative to the equilibrium position after t seconds. Assume that at $t = 0$, the object begins at its equilibrium position with a velocity of 0. Set up and solve the differential equation that determines $x(t)$.

$$mx''(t) + bx'(t) + kx(t) = F(t) \quad \text{so in this case,} \quad x''(t) + 4x'(t) + 4x(t) = 3te^{-2t}$$

$$(D+2)^2x(t) = 3te^{-2t}$$

The homogeneous solution is $x_H(t) = c_1e^{-2t} + c_2te^{-2t}$. The Method of Undetermined Coefficients can be used to find that the particular solution is $x_p(t) = \frac{1}{2}t^3e^{-2t}$. The general solution is therefore:

$$x(t) = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{2}t^3e^{-2t}$$

The initial conditions imply that $c_1 = 0$ and $c_2 = 0$ so the solution to this problem is:

$$x(t) = \frac{1}{2}t^3e^{-2t}$$

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4. An object of mass 2 kilograms is attached to a spring with a spring constant $k = 8$ newtons/meter. Assume that there is no damping force due to friction, but there is a time dependent force of $F(t) = 12\cos 2t$. Assume also that the object begins with an initial velocity of 0 and an initial displacement of 0. Determine the displacement of the object as a function of time by solving an appropriate differential equation.

$$2y'' + 8y = 12\cos 2t \qquad \text{where } y(0) = y'(0) = 0$$

$$y'' + 4y = 6\cos 2t$$

The homogeneous solution is a linear combination of $\cos 2t$ and $\sin 2t$. The particular solution must be a linear combination of $t\cos 2t$ and $t\sin 2t$. We can use the Method of Undetermined Coefficients to solve for b_1 and b_2 so that:

$$(D^2+4)(b_1 t\cos 2t + b_2 t\sin 2t) = 6\cos 2t$$

$$-4b_1\sin 2t + 4b_2\cos 2t = 6\cos 2t$$

Therefore, $b_1 = 0$ and $b_2 = \frac{3}{2}$. The general solution must be:

$$y(t) = a_1\cos 2t + a_2\sin 2t + \frac{3}{2}t\sin 2t$$

The initial conditions imply that $a_1 = 0$ and $a_2 = 0$ so the solution is:

$$y(t) = \frac{3}{2}t\sin 2t = 3t\sin t\cos t$$

5. An electric circuit has an electromotive force given by $\mathcal{E}(t) = 60\cos 5t$ volts, an inductor of 1 henry, a resistor of 6 ohms and a capacitor of 0.04 farads. Let $Q(t)$ be the charge on the capacitor. $Q(t)$ will solve the differential equation:

$$\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 25Q = 60\cos 5t$$

If $Q(t)$ satisfies the initial conditions $Q(0) = 0$ and $Q'(0) = 6$, find the formula for $Q(t)$.

The homogeneous solution is a linear combination of $e^{-3t}\cos 4t$ and $e^{-3t}\sin 4t$. The particular solution must have the form $A\cos 5t + B\sin 5t$. If we substitute $A\cos 5t + B\sin 5t$ into the differential equation, we obtain $A = 0$ and $B = 2$. The general solution of the differential equation is therefore:

$$Q(t) = ae^{-3t}\cos 4t + be^{-3t}\sin 4t + 2\sin 5t$$

If we now impose the initial conditions, we obtain $a = 0$ and $b = -1$, so:

$$Q(t) = -e^{-3t}\sin 4t + 2\sin 5t$$
