Assignment 15. *Variation of Parameters*

(1)
$$y'' + y = \sin x \cos x$$
$$y = v_1 \cos x + v_2 \sin x$$
$$\text{Wronskian Determinant: } \mathcal{W} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$
$$\frac{dv_1}{dx} = \begin{vmatrix} 0 & \sin x \\ \sin x \cos x & \cos x \end{vmatrix} = -\sin^2 x \cos x \quad \text{so } v_1 = -\frac{1}{3}\sin^3 x + C_1$$
$$\frac{dv_2}{dx} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \cos x \end{vmatrix} = \cos^2 x \sin x \quad \text{so } v_1 = -\frac{1}{3}\cos^3 x + C_2$$
$$y = C_1 \cos x + C_2 \sin x - \frac{1}{3}\sin x \cos x(\sin^2 x + \cos^2 x) = C_1 \cos x + C_2 \sin x - \frac{1}{3}\sin x \cos x$$
$$\text{The answer can also be written as :} y = C_1 \cos x + C_2 \sin x - \frac{1}{6}\sin 2x$$

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(2)
$$(D^2 - D)y = \frac{e^{2x}}{e^{2x}+1}$$
$$y = v_1 + v_2 e^x$$
$$\frac{dv_1}{dx} = \frac{\begin{vmatrix} 0 & e^x \\ \frac{e^{2x}}{e^{2x}+1} & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = -\frac{e^{2x}}{e^{2x}+1}$$
$$\frac{dv_2}{dx} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & \frac{e^{2x}}{e^{2x}+1} \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{e^x}{e^{2x}+1}$$
$$v_1 = \int -\frac{e^{2x}}{e^{2x}+1}\,dx = -\frac{1}{2}\ln\left(e^{2x}+1\right) + c_1$$
$$v_2 = \int \frac{e^x}{e^{2x}+1}\,dx = \tan^{-1}\left(e^x\right) + c_2$$
$$y = c_1 + c_2 e^x - \frac{1}{2}\ln\left(e^{2x}+1\right) + e^x \tan^{-1}\left(e^x\right)$$

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(3)
$$y'' - 2y' + y = x^2 e^x$$
$$y = v_1 e^x + v_2 x e^x$$
$$\text{Wronskian Determinant: } \mathcal{W} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x}$$
$$\frac{dv_1}{dx} = e^{-2x}\begin{vmatrix} 0 & xe^x \\ x^2 e^x & (x+1)e^x \end{vmatrix} = -x^3 \quad \text{so} \quad v_1 = \int -x^3\,dx = -\frac{1}{4}x^4 + C_1$$
$$\frac{dv_2}{dx} = e^{-2x}\begin{vmatrix} e^x & 0 \\ e^x & x^2 e^x \end{vmatrix} = x^2 \quad \text{so} \quad v_2 = \int x^2\,dx = \frac{1}{3}x^3 + C_2$$
$$y = \left(-\frac{1}{4}x^4 + C_1\right)e^x + \left(\frac{1}{3}x^3 + C_2\right)xe^x = C_1 e^x + C_2 x e^x + \frac{1}{12}x^4 e^x$$

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4. The Method of Variation of Parameters can be used to find the general solution of the following differential equation:

$$\left(D - \frac{1}{2}\right)^2 y = x^{-2} e^{\frac{x}{2}}$$

The solution has the form:

$$y = v_1(x) \cdot e^{\frac{x}{2}} + v_2(x) \cdot x e^{\frac{x}{2}}$$

where $v_1(x)$ and $v_2(x)$ are functions. Calculate the function $v_1(x)$

The homogeneous solutions are $y_1 = e^{\frac{x}{2}}$ and $y_2 = xe^{\frac{x}{2}}$. We first find the Wronskian determinant:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\frac{x}{2}} & xe^{\frac{x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & \left(1 + \frac{x}{2}\right) e^{\frac{x}{2}} \end{vmatrix} = e^x$$

$$\frac{dv_1}{dx} = \frac{\begin{vmatrix} 0 & xe^{\frac{x}{2}} \\ x^{-2}e^{\frac{x}{2}} & \left(1 + \frac{x}{2}\right) e^{\frac{x}{2}} \end{vmatrix}}{e^x} = \frac{-x^{-1}e^x}{e^x} = -\frac{1}{x} \quad \text{so} \quad v_1 = -\int \frac{1}{x}\, dx = -\ln x + C_1$$
