Assignment 16. *Introduction to Laplace Transforms*

1. The Laplace transform of a function $f(t)$ is defined to be: $\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$. Use this integral to calculate the Laplace transforms of each of the following functions:

a) $t+1$ b) $(t+1)e^{-4t}$

a)
$$\begin{aligned}\mathcal{L}(t+1) &= \int_0^\infty e^{-st}(t+1)\,dt \\ &= -(t+1)\cdot\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s} + \frac{1}{s^2}\end{aligned}$$

b)
$$\mathcal{L}\left((t+1)e^{-4t}\right) = \int_0^\infty (t+1)e^{-4t}e^{-st}\,dt = \int_0^\infty (t+1)e^{-ut}\,dt \text{ where } u = (s+4)$$

This is the same as part (a) except the parameter is u instead of s. The answer is therefore:

$$\mathcal{L}\left((t+1)e^{-4t}\right) = \frac{1}{u} + \frac{1}{u^2} = \frac{1}{s+4} + \frac{1}{(s+4)^2}$$

2. Use a table of Laplace transforms as well as any relevant properties of Laplace transforms to find the following transforms:

a)
$$\mathcal{L}\left(e^{6t} - e^{-6t}\right) = \mathcal{L}\left(e^{6t}\right) - \mathcal{L}\left(e^{-6t}\right) = \frac{1}{s-6} - \frac{1}{s+6} = \frac{12}{s^2-36}$$

b)
$$\mathcal{L}\left(t^2e^{-t}\right) = \frac{2}{(s+1)^3}$$

c)
$$\mathcal{L}(\cos 2t) = \frac{s}{s^2+4}$$

d)
$$\mathcal{L}\left(e^{3t}\cos 2t\right) = \frac{s-3}{(s-3)^2+4}$$

e)
$$\mathcal{L}\left(e^{-t}\cosh t\right) = \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right)$$

3. Use any legitimate method to calculate the Laplace transform of the following function:

$$f(t) = \begin{cases} te^t & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t \end{cases}$$

$$\begin{aligned}\text{Ł}f(t) = \int_0^1 te^te^{-st}\,dt = \int_0^1 te^{(1-s)t}\,dt &= \frac{1}{1-s}te^{(1-s)t}\Big|_0^1 - \int_0^1 \frac{1}{1-s}e^{(1-s)t}\,dt \\ &= \frac{1}{1-s}e^{1-s} - \frac{1}{(1-s)^2}e^{(1-s)t}\Big|_0^1 = \frac{1}{1-s}e^{1-s} - \frac{1}{(1-s)^2}e^{1-s} + \frac{1}{(1-s)^2}\end{aligned}$$

4. Calculate the inverse Laplace transforms of each of the following:

a)
$$\mathcal{L}^{-1}\left(\frac{6}{(s-2)^2}\right) = 6\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) = 6te^{2t}$$

b)
$$\mathcal{L}^{-1}\left(\frac{1}{(2s+1)^2}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2}\right) = \frac{1}{4}te^{-t/2}$$

c)
$$\mathcal{L}^{-1}\left(\frac{8}{(s+2)(s+4)}\right) = \mathcal{L}^{-1}\left(\frac{4}{s+2} - \frac{4}{s+4}\right) = 4e^{-2t} - 4e^{-4t}$$

d)
$$\mathcal{L}^{-1}\left(\frac{3}{(s^2+1)(s^2+4)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) = \sin t - \frac{1}{2}\sin 2t$$

e)
$$\mathcal{L}^{-1}\left(\frac{1}{s^4+s^3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1}\right) = 1 - t + \frac{1}{2}t^2 - e^{-t}$$