Assignment 17. *Using Laplace Transforms to Solve Differential Equations*

1. Problem 1 is deleted

2.

$$y'' - 4y = -6e^t, \qquad y(0) = 3\ y'(0) = 0$$
$$\mathcal{L}(y'') - 4\mathcal{L}(y) = \frac{-6}{s-1}$$
$$s^2\mathcal{L}(y) - 3s - 4\mathcal{L}(y) = \frac{-6}{s-1}$$
$$\left(s^2 - 4\right)\mathcal{L}(y) = 3s - \frac{6}{s-1} = \frac{3s^2 - 3s - 6}{s-1}$$
$$\mathcal{L}(y) = \frac{3s^2 - 3s - 6}{(s^2-4)(s-1)} = \frac{1}{s+2} + \frac{2}{s-1} \qquad \text{(partial fractions decomposition)}$$
$$y = \mathcal{L}^{-1}\left(\frac{1}{s+2} + \frac{2}{s-1}\right) = e^{-2t} + 2e^t$$

3.

$$y'' + 2y' = 2t + 3, \qquad y(0) = 0,\ y'(0) = 1$$
$$Ły'' + 2Ły' = \frac{2}{s^2} + \frac{3}{s} \quad \text{which reduces to} \quad (s^2+2s)Ły - 1 = \frac{2}{s^2} + \frac{3}{s}$$
$$s(s+2)Ły = \frac{s^2+3s+2}{s^2} = \frac{(s+1)(s+2)}{s^2}$$
$$Ły = \frac{(s+1)(s+2)}{s^3(s+2)} = \frac{s+1}{s^3} = \frac{1}{s^2} + \frac{1}{s^3}$$
$$y = t + \frac{1}{2}t^2$$

4.

$$y'' + 8y' + 25y = 100, \qquad y(0) = 2,\ y'(0) = 20$$
$$(s^2+8s+25)\mathcal{L}(y) - 2s - 36 = \frac{100}{s}$$
$$\mathcal{L}(y) = \frac{2s^2+36s+100}{s(s^2+8s+25)} = \frac{4}{s} + \frac{4-2s}{s^2+8s+25} = \frac{4}{s} - 2\left(\frac{s+4}{(s+4)^2+9}\right) + 4\left(\frac{3}{(s+4)^2+9}\right)$$
$$y(t) = 4 - 2e^{-4t}\cos 3t + 4e^{-4t}\sin 3t$$

5.

$$y'' - 4y' + 4y = t^3e^{2t} \qquad y(0) = 0,\ y'(0) = 0$$
$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 4\mathcal{L}(y) = \frac{3!}{(s-2)^4}$$
$$s^2\mathcal{L}(y) - sy(0) - y'(0) - 4(s\mathcal{L}(y) - y(0)) + 4\mathcal{L}(y) = \frac{6}{(s-2)^4}$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$

$$(s^2 - 4s + 4)\mathcal{L}(y) = \frac{6}{(s-2)^4}$$
$$(s-2)^2\mathcal{L}(y) = \frac{6}{(s-2)^4}$$
$$\mathcal{L}(y) = \frac{6}{(s-2)^6} = 6 \cdot \frac{1}{120} \cdot \frac{5!}{(s-2)^6} = \frac{1}{20}\mathcal{L}\left(t^5e^{2t}\right)$$
$$y(t) = \frac{1}{20}t^5e^{2t}$$