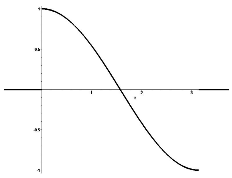
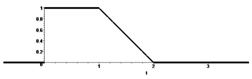
Assignment 18

1. Graph each of the following functions. The symbol $\mathcal{U}$ refers to the *unit step function.*

a. $f(t) = (\cos t) \cdot (\mathcal{U}(t) - \mathcal{U}(t-\pi))$



b. $g(t) = \mathcal{U}(t) - \mathcal{U}(t-1) + (2-t) \cdot (\mathcal{U}(t-1) - \mathcal{U}(t-2))$



2. Express the following $f(t)$ in terms of the unit step function $\mathcal{U}$ and then calculate $\mathcal{L}(f)$. You may use a table of Laplace transforms.

$$f(t) = \begin{cases} t & \text{when } 0 \le t < 1 \\ e^{1-t} & \text{when } t \ge 1 \end{cases}$$

$$f(t) = t(\mathcal{U}(t) - \mathcal{U}(t-1)) + e^{1-t}\mathcal{U}(t-1)$$

$$\mathcal{L}(f) = \frac{1}{s^2} + e^{-s}\left(-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s+1}\right)$$

3. The electric charge $y(t)$ in an LC series circuit is governed by the initial value problem:

$$y''(t) + 4y(t) = 3\sin t - 3\sin t \cdot \mathcal{U}(t-2\pi), \qquad y(0) = 1,\ y'(0) = 3$$

where $\mathcal{U}$ is the *unit step function.*
Determine the charge as a function of time t.

$$(s^2+4)\mathcal{L}(y) - s - 3 = \frac{3}{s^2+1} - \frac{e^{-2\pi s} \cdot 3}{s^2+1}$$

$$\begin{aligned} \mathcal{L}(y) &= \frac{s}{s^2+4} + \frac{3}{s^2+4} + \frac{3}{(s^2+1)(s^2+4)} - \frac{3e^{-2\pi s}}{(s^2+4)(s^2+1)} \\ &= \frac{s}{s^2+4} + \frac{3}{s^2+4} + \frac{1}{s^2+1} - \frac{1}{s^2+4} - e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) \end{aligned}$$

$$\mathcal{L}(y) = \frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{1}{s^2+1} - e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$\begin{aligned} y(t) &= \cos 2t + \sin 2t + \sin t - \sin(t-2\pi)\mathcal{U}(t-2\pi) + \frac{1}{2}\sin(2(t-2\pi))\mathcal{U}(t-2\pi) \\ &= \cos 2t + \sin 2t + \sin t - \sin t \cdot \mathcal{U}(t-2\pi) + \frac{1}{2}\sin 2t \cdot \mathcal{U}(t-2\pi) \end{aligned}$$

4. **Problem 4 is deleted**.

5. At $t = 0$, a tank contains 16 liters of a brine solution with 10 grams of salt dissolved in it. Then, we start draining fluid out at 8 liters per minute. For $0 \le t \le 2$, valve A is open and only pure water is being pumped in at 8 liters per minute. Then, for $t > 2$, valve A is closed and valve B is opened and salt solution containing $\frac{1}{2}$ grams of salt per liter is pumped in at 8 liters per minute.

a. Let $x(t)$ be the number of grams of salt in the tank after t minutes. Set up an appropriate differential equation for $x(t)$, making appropriate use of the unit step function $\mathcal{U}$.

The rate at which salt is leaving is $\frac{8 \text{ liters}}{\text{min}} \cdot \frac{x \text{ grams}}{16 \text{ liters}} = \frac{x}{2}$.

The rate at which salt is entering is 0 for $t \le 2$ and $\frac{1}{2} \frac{\text{grams}}{\text{liter}} \cdot \frac{8 \text{ liters}}{\text{min}} = 4$ for $t > 2$. Therefore, the rate in can be written as $4\mathcal{U}(t-2)$

$$\frac{dx}{dt} = 4\mathcal{U}(t-2) - \frac{1}{2}x$$

b. Solve the differential equation using the method of Laplace transforms. Your final answer should be expressed in terms of the unit step function.

$$\mathcal{L}(x') = \mathcal{L}\left(4\mathcal{U}(t-2) - \frac{1}{2}x\right)$$

$$\left(s + \frac{1}{2}\right)\mathcal{L}(x) - 10 = \frac{4}{s}e^{-2s}$$

$$\mathcal{L}(x) = \frac{10}{s+\frac{1}{2}} + \frac{4}{s\left(s+\frac{1}{2}\right)}e^{-2s} = \frac{10}{s+\frac{1}{2}} + \left(\frac{8}{s} - \frac{8}{s+\frac{1}{2}}\right)e^{-2s} = \mathcal{L}\left(10e^{-t/2}\right) + \mathcal{L}\left(8 - 8e^{-t/2}\right)e^{-2s}$$

$$x(t) = 10e^{-t/2} + \left(8 - 8e^{-(t-2)/2}\right)\mathcal{U}(t-2) = 10e^{-t/2} + 8\left(1 - e^{1-t/2}\right)\mathcal{U}(t-2)$$