Assignment 19. *Series Solutions*

(1) $y'' - y = 0$ where $y(0) = y'(0) = 1$

$$\sum n(n-1)a_n x^{n-2} - \sum a_n x^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} - a_n\right) x^n = 0$$

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \qquad a_0 = 1 \qquad a_1 = 1$$

Repeated substitution yields:

$$a_n = \frac{1}{n!}$$

$$y = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Note: This is the same as $y = e^x$

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(2) $y'' - 2xy' - 2y = 0$ where $y(0) = 1$, $y'(0) = 0$

$$\sum n(n-1)a_n x^{n-2} - 2x \sum n a_n x^{n-1} - 2 \sum a_n x^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} - 2(n+1)a_n\right) x^n = 0$$

$$a_{n+2} = \frac{2a_n}{n+2} \qquad a_0 = 1 \qquad a_1 = 0$$

Repeated substitution yields:

$$a_2 = 1 \qquad a_3 = 0 \qquad a_4 = \frac{1}{2} \qquad a_5 = 0 \qquad a_6 = \frac{1}{3!} \qquad a_7 = 0 \qquad a_8 = \frac{1}{4!} \qquad \ldots \qquad \text{etc.}$$

$$y = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

Note: This is the same as $y = e^{x^2}$

3. $(1-2x)y'' - 4y' = 0$ where $y(0) = 0$ and $y'(0) = 2$

$$(1-2x)\sum (n+2)(n+1)a_{n+2}x^n - 4\sum (n+1)a_{n+1}x^n = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n - \sum 2(n+2)(n+1)a_{n+2}x^{n+1} - 4\sum (n+1)a_{n+1}x^n = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n - \sum 2(n+1)(n)a_{n+1}x^n - 4\sum (n+1)a_{n+1}x^n = 0$$

$$\sum \left[(n+2)(n+1)a_{n+2} - 2(n+1)(n)a_{n+1} - 4(n+1)a_{n+1}\right] x^n = 0$$

$$(n+2)(n+1)a_{n+2} - 2(n+1)(n)a_{n+1} - 4(n+1)a_{n+1} = 0 \quad \text{for all } n$$

$$(n+2)a_{n+2} - (2n+4)a_{n+1} \quad \text{for } n \geq 0$$

$$a_{n+2} = 2a_{n+1}$$

The initial conditions imply that $a_0 = 0$ and $a_1 = 2$. The remaining coefficients are determined by the recurrence relation $a_{n+2} = 2a_{n+1}$

$$a_2 = 2a_1 = 2^2\,, \quad a_3 = 2a_2 = 2^3 \quad a_4 = 2a_3 = 2^4 \quad \cdots \quad a_n = 2^n \quad (\text{for } n \geq 1)$$

$$y = \sum_{n=1}^{\infty} 2^n x^n$$

Note that this is a geometric series that simplifies to:

$$y = \frac{2x}{1-2x}$$

4. The following differential equation has a solution of the form $\sum a_n x^n$. Calculate only the coefficients a_2, a_3 and a_4.

$$\left(1+x^2\right) y'' - 2y = 0 \quad \text{where } y(0) = y'(0) = 1$$

Substitute $y = \sum a_n x^n$ and $y'' = \sum n(n-1)a_n x^{n-2}$. Note that we are already given that $a_0 = a_1 = 1$

$$\sum n(n-1)a_n x^{n-2} + x^2 \sum n(n-1)a_n x^{n-2} - 2\sum a_n x^n = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n + \sum n(n-1)a_n x^n - 2\sum a_n x^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} + (n^2-n-2)a_n\right) x^n = 0$$

$$a_{n+2} = -\frac{(n^2-n-2)}{(n+2)(n+1)}a_n = -\frac{(n+1)(n-2)}{(n+2)(n+1)}a_n = -\frac{(n-2)}{n+2}a_n$$

$$a_2 = -\frac{(0-2)}{0+2}a_0 = a_0 = 1$$

$$a_3 = -\frac{(1-2)}{1+2}a_1 = \frac{1}{3}a_1 = \frac{1}{3}$$

$$a_4 = -\frac{(2-2)}{2+2}a_2 = 0$$