Assignment 20. *Eigenvalues, Eigenvectors and Matrix Differential Equations*

(1) For each of the following matrices, calculate all eigenvalues and corresponding eigenvectors.

a

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$det(A - \lambda I) = (1-\lambda)(3-\lambda) = 0 \text{ when } \lambda = 1,\, 3$$

$$\text{For } \lambda = 1, \text{ the eigenvector is } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{For } \lambda = 3, \text{ the eigenvector is } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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b

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}$$

$$det(A - \lambda I) = \lambda(\lambda+5) = 0 \text{ when } \lambda = 0,\, -5$$

$$\text{For } \lambda = 0, \text{ the eigenvector is } \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \text{For } \lambda = -5, \text{ the eigenvector is } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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c

$$\begin{pmatrix} 3 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$det(A - \lambda I) = (2-\lambda)(\lambda-5)(\lambda+1) = 0 \text{ when } \lambda = -1,\, 2,\, 5$$

$$\text{For } \lambda = -1, \text{ the eigenvector is } \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \qquad \text{For } \lambda = 2, \text{ the eigenvector is } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{For } \lambda = 5, \text{ the eigenvector is } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

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d

$$\begin{pmatrix} 0 & 1 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & -3 \end{pmatrix}$$

$$det(A - \lambda I) = \lambda(2-\lambda)(\lambda+3) = 0 \text{ when } \lambda = 0,\, 2,\, -3$$

$$\text{For } \lambda = 0, \text{ the eigenvector is } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{For } \lambda = 2, \text{ the eigenvector is } \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\text{For } \lambda = -3, \text{ the eigenvector is } \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$$

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(2) Find the general solution of the differential equation

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \qquad \text{where} \quad A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{\mathbf{x}} = \begin{pmatrix} c_1 e^{4t} \\ c_2 e^{3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}$$

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(3) Find the general solution of the differential equation

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \qquad \text{where} \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$det(A - \lambda I) = (\lambda - 4)(\lambda + 2) = 0 \text{ when } \lambda = 4, -2$$

For $\lambda = 4,$ the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ For $\lambda = -2,$ the eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\vec{\mathbf{x}}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

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(4) Find the solution of the differential equation

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \qquad \vec{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{where} \quad A = \begin{pmatrix} -7 & 24 \\ -2 & 7 \end{pmatrix} \quad \text{and} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$det(A - \lambda I) = (\lambda - 1)(\lambda + 1) = 0 \text{ when } \lambda = 1, -1$$

For $\lambda = 1,$ the eigenvector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ For $\lambda = -1,$ the eigenvector is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$\vec{\mathbf{x}}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-t}$$

The initial condition implies $c_1 = -1$ and $c_2 = 1$

$$\vec{\mathbf{x}}(t) = -\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-t}$$

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(5) Convert the ordinary differential equation

$$y'' - 4y' + 3y = 0$$

to a matrix differential of the form

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$$

Solve the matrix differential equation.

$$\text{Let } \vec{\mathbf{x}}(t) = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \vec{\mathbf{x}}$$

$$det(A - \lambda I) = (\lambda - 1)(\lambda - 3) = 0 \text{ when } \lambda = 1, 3$$

For $\lambda = 1,$ the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ For $\lambda = 3,$ the eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\vec{\mathbf{x}}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}$$

$y(t)$ is the first coordinate of $\vec{\mathbf{x}}(t)$, so $y(t) = c_1 e^t + c_2 e^{3t}$