**Laplace Transform Solution of Differential Equations**
**Jacobs**

At the end of the lesson on inverse Laplace transforms, we were able to use it to solve a simple first order differential equation. The purpose of this lesson is to generalize the method to higher order equations. To review the method, let's start with the following equation:

$$y' = 2y + e^{2t} \qquad \text{where } y(0) = 0$$

The method of Laplace transforms is a three step procedure. The first step is to take the Laplace transform of both sides of the equation and use the identity $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$.

$$\mathcal{L}(y') = \mathcal{L}\left(2y + e^{2t}\right)$$

$$s\mathcal{L}(y) - sy(0) = 2\mathcal{L}(y) + \mathcal{L}\left(e^{2t}\right)$$

$$s\mathcal{L}(y) - 0 = 2\mathcal{L}(y) + \frac{1}{s-2}$$

Notice that what we have done is to start with a differential equation involving $y(t)$ and transform it to an *algebraic* equation involving $\mathcal{L}(y)$. Algebraic equations are easier to solve, so let's do it. This leads us to the second step of the process. Solve for $\mathcal{L}(y)$

$$s\mathcal{L}(y) = 2\mathcal{L}(y) + \frac{1}{s-2}$$

$$s\mathcal{L}(y) - 2\mathcal{L}(y) = \frac{1}{s-2}$$

$$(s-2)\mathcal{L}(y) = \frac{1}{s-2}$$

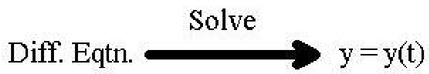
$$\mathcal{L}(y) = \frac{1}{(s-2)^2}$$

Now, for the final step. We are looking for $y = y(t)$. Instead we have found its Laplace transform. All that's left to do is find the inverse Laplace transform.

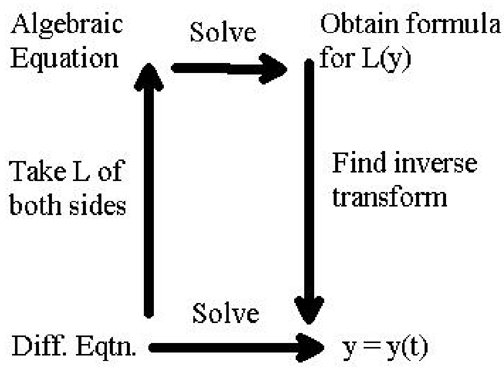
$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) = te^{2t}$$

A quick word about notation. A common notation for the Laplace transform is to user $Y(s)$ instead of $\mathcal{L}(y)$ when doing calculations. Thus, the equation we just solved would have been $sY(s) = 2Y(s) + \frac{1}{s-2}$ which we would then solve to obtain $Y(s) = \frac{1}{(s-2)^2}$. It all comes to the same thing. The danger is that students are usually racing to get to the answer and they'll start writing $Y$ instead of $Y(s)$ and soon $Y$ starts looking like $y$ and the confusion can lead to errors. In this lesson, I'll stick to $\mathcal{L}(y)$ notation.

Let's summarize the procedure. Remember, the ultimate goal is to solve the differential equation and obtain the solution $y = y(t)$.




With Laplace transforms, we go through three steps in a somewhat roundabout way to get to the answer.




The big advantage is that we are solving an algebraic equation instead of a differential equation. The only step that is potentially difficult is the final step when we need the inverse Laplace transform.

Let's try this out on a second order homogeneous differential equation. We need to extend the identity $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$. Simply replace $f$ by $y'$ and the identity becomes

$$\begin{aligned}\mathcal{L}(y'') &= s\mathcal{L}(y') - y'(0)\\ &= s(s\mathcal{L}(y) - y(0)) - y'0)\\ &= s^2\mathcal{L}(y) - sy(0) - y'(0)\end{aligned}$$

We are ready to solve a second order equation.

**Example:** Solve the following differential equation.

$$y'' + 2y = 0 \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

We start by taking the Laplace transform of both sides of the equation.

$$\mathcal{L}(y'') + 2\mathcal{L}(y) = \mathcal{L}(0)$$

$\mathcal{L}(0) = \int_0^\infty 0e^{-st}\,dt = 0$. Now, use $\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - 0 - 1$

$$s^2\mathcal{L}(y) - 1 + 2\mathcal{L}(y) = 0$$

Solve for $\mathcal{L}(y)$

$$\mathcal{L}(y) = \frac{1}{s^2+2}$$

If we can find the inverse Laplace transform, then we will have our solution. This is usually the hard part. In this case, we look at a table of Laplace transforms and discover the following formula:

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

so, if $\omega = \sqrt{2}$ we'll have $\mathcal{L}\left(\sin\sqrt{2}t\right) = \frac{\sqrt{2}}{s^2+2}$ and we are ready to get the inverse transform.

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+2}\right) = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{s^2+2}\right) = \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

**Example:** Solve the following differential equation.

$$y'' + 2y' + 2y = 0 \qquad \text{where } y(0) = 1 \text{ and } y'(0) = -1$$

We begin by taking the Laplace transform of both sides and using the identities $\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - s + 1$ and also $\mathcal{L}(y') = s\mathcal{L}(y) - y(0) = s\mathcal{L}(y) - 1$

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(0)$$

$$s^2\mathcal{L}(y) - s + 1 + 2(s\mathcal{L}(y) - 1) + 2\mathcal{L}(y) = 0$$

Solve for $\mathcal{L}(y)$

$$(s^2 + 2s + 2)\mathcal{L}(y) = s + 1$$

$$\mathcal{L}(y) = \frac{s+1}{s^2 + 2s + 2}$$

The inverse Laplace transform isn't bad if you complete the square in the denominator and refer to the formula $\mathcal{L}\left(e^{\lambda t}\cos\omega t\right) = \frac{s-\lambda}{(s-\lambda)^2+\omega^2}$

$$y = \mathcal{L}^{-1}\left(\frac{s+1}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) = e^{-t}\cos t$$

The technique we have been using is easily extended to differential equations of third order or higher.

$$\begin{aligned}\mathcal{L}(y''') &= s\mathcal{L}(y'') - y''(0) \\ &= s(s^2\mathcal{L}(y) - sy(0) - y'(0)) - y''(0) \\ &= s^3\mathcal{L}(y) - s^2y(0) - sy'(0) - y''(0)\end{aligned}$$

Example: Solve the following differential equation:

$$y''' - 3y'' + 3y' - y = 0 \qquad \text{where } y(0) = y'(0) = 0 \text{ and } y''(0) = 1$$

We take the Laplace transform of both sides.

$$\mathcal{L}(y''') - 3\mathcal{L}(y'') + 3\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(0)$$

$$s^3\mathcal{L}\left(y\right) - s^2 \cdot 0 - s \cdot 0 - 1 - 3(s^2\mathcal{L}\left(y\right) - s \cdot 0 - 0) + 3(s\mathcal{L}\left(y\right) - 0) - \mathcal{L}\left(y\right) = 0$$

$$(s^3 - 3s^2 + 3s - 1)\mathcal{L}\left(y\right) = 1$$

Solve for $\mathcal{L}\left(y\right)$ and then take the inverse Laplace transform.

$$\mathcal{L}\left(y\right) = \frac{1}{s^3 - 3s^2 + 3s - 1} = \frac{1}{(s-1)^3}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^3}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{(s-1)^3}\right) = \frac{1}{2}e^t t^2$$

The last couple of examples were homogeneous differential equations. We can use the same technique on nonhomogeneous differential equations, although the algebra can become a little more complicated.

**Example:**

Solve the following differential equation:

$$(D^2 - 4)y = 3e^t \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

As usual, we begin by taking $\mathcal{L}$ of both sides:

$$\mathcal{L}\left(y'' - 4y\right) = \mathcal{L}\left(3e^t\right)$$

$$s^2\mathcal{L}\left(y\right) - s \cdot 0 - 1 - 4\mathcal{L}\left(y\right) = \frac{3}{s-1}$$

$$(s^2 - 4)\mathcal{L}\left(y\right) = 1 + \frac{3}{s-1} = \frac{s+2}{s-1}$$

$$\mathcal{L}\left(y\right) = \frac{s+2}{(s-1)(s^2-4)} = \frac{s+2}{(s-1)(s-2)(s+2)} = \frac{1}{(s-1)(s-2)}$$

A partial fractions decomposition will simplify the right hand side and we will be able to get the inverse Laplace transform.

$$\mathcal{L}\left(y\right) = \frac{1}{s-2} - \frac{1}{s-1}$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{1}{s-1}\right) = e^{2t} - e^t$$

Once again, we needed partial fractions to complete the problem. Partial fractions is turning out to be more important in MA 345 than it was in MA 242. The partial fractions calculation can get more complicated in other equations.

**Example:**

Solve the following differential equation:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 2e^t + 2e^{-t} \qquad \text{where } y(0) = y'(0) = 0$$

If we take the Laplace transform of both sides, we get:

$$s^2\mathcal{L}(y) + s\mathcal{L}(y) = \frac{2}{s-1} + \frac{2}{s+1} = \frac{4s}{(s-1)(s+1)}$$

$$\mathcal{L}(y) = \frac{4}{(s-1)(s+1)^2}$$

The partial fractions decomposition will have the general form:

$$\frac{4}{(s-1)(s+1)^2} = \frac{a}{s-1} + \frac{b}{s+1} + \frac{c}{(s+1)^2}$$

If you solve for $a$, $b$ and $c$, you will get $a = 1$, $b = -1$ and $c = -2$. Therefore,

$$\mathcal{L}(y) = \frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2}$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2}\right) = e^t - e^{-t} - 2te^{-t}$$

**Example:**

Solve the following differential equation:

$$y'' + 2y' + 2y = 4 + 4t \qquad \text{where } y(0) = 3 \text{ and } y'(0) = 2$$

Take $\mathcal{L}$ of both sides:

$$s^2\mathcal{L}(y) - 3s - 2 + 2(s\mathcal{L}(y) - 3) + 2\mathcal{L}(y) = \frac{4}{s} + \frac{4}{s^2}$$

We can solve for $\mathcal{L}(y)$, though it takes a little more algebra this time.

$$\mathcal{L}(y) = \frac{3s^3 + 8s^2 + 4s + 4}{s^2(s^2 + 2s + 2)}$$

This has a partial fractions decomposition:

$$\mathcal{L}(y) = \frac{2}{s^2} + \frac{6 + 3s}{s^2 + 2s + 2}$$

We did a problem earlier where the denominator was $s^2 + 2s + 2$ and we found that completing the square helped.

$$\mathcal{L}(y) = \frac{2}{s^2} + \frac{6 + 3s}{(s+1)^2 + 1}$$

This is when we consult our table of Laplace transforms. Two transform formulas that seem relevant are:

$$\mathcal{L}\left(e^{\lambda t} \sin \omega t\right) = \frac{\omega}{(s-\lambda)^2 + \omega^2} \qquad \mathcal{L}\left(e^{\lambda t} \cos \omega t\right) = \frac{s-\lambda}{(s-\lambda)^2 + \omega^2}$$

If we substitute $\lambda = -1$ and $\omega = 1$, these formulas become:

$$\mathcal{L}\left(e^{-t} \sin t\right) = \frac{1}{(s+1)^2 + 1} \qquad \mathcal{L}\left(e^{-t} \cos t\right) = \frac{s+1}{(s+1)^2 + 1}$$

It takes just a little bit of algebraic juggling to get $\mathcal{L}(y)$ in terms of these expressions.

$$\begin{aligned} \mathcal{L}(y) &= \frac{2}{s^2} + \frac{6 + 3s}{(s+1)^2 + 1} \\ &= \frac{2}{s^2} + \frac{3 + 3(s+1)}{(s+1)^2 + 1} \\ &= \frac{2}{s^2} + \frac{3}{(s+1)^2 + 1} + \frac{3(s+1)}{(s+1)^2 + 1} \end{aligned}$$

Finally, we are able to take the inverse Laplace transform.

$$y(t) = 2t + 3e^{-t} \sin t + 3e^{-t} \cos t$$