Computation of Laplace Transforms
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We have one last method that is used to solve linear differential equations called the Method of Laplace Transforms. This is a method that is frequently used in engineering courses and it is sufficiently difficult that we will need a couple of weeks to study it. The method relies on improper integrals, so let's do a quick review example, starting with $\int_0^\infty e^{-7t}\,dt$. As you learned in Calculus II, this is defined in terms of limits:

$$\int_0^\infty e^{-7t}\,dt = \lim_{T\to\infty}\int_0^T e^{-7t}\,dt = \lim_{T\to\infty}\frac{1}{7}\left(1-e^{-7T}\right) = \frac{1}{7}$$

Now, let's change the integral and calculate $\int_0^\infty e^{-7t}e^{-st}\,dt$, where s is a constant.

$$\int_0^\infty e^{-7t}e^{-st}\,dt = \lim_{T\to\infty}\int_0^T e^{(-s-7)t}\,dt = \lim_{T\to\infty}\frac{1}{-s-7}\left(e^{-(s+7)T}-1\right)$$

This integral will only converge if $s+7>0$ and if that is the case then $e^{-(s+7)T}\longrightarrow 0$ when $T\longrightarrow\infty$. We can now complete the integral calculation:

$$\int_0^\infty e^{-7t}e^{-st}\,dt = \lim_{T\to\infty}\frac{1}{-s-7}\left(e^{-(s+7)T}-1\right) = \frac{1}{-s-7}(0-1) = \frac{1}{s+7}$$

We have just done our very first Laplace transform problem. The Laplace transform of $f(t)$ is defined to be:

$$\mathcal{L}\left(f(t)\right) = \int_0^\infty f(t)e^{-st}\,dt$$

So, according to the calculation we have just done, the Laplace transform of e^{7t} is $\frac{1}{s+7}$.

$$\mathcal{L}\left(e^{-7t}\right) = \frac{1}{s+7}$$

We have just "transformed" a function of t into a different function of s (hence the terminology Laplace *transform*). In general, we write this as $\mathcal{L}\left(f(t)\right) = F(s)$. If we started with $F(s)$ and we wanted to find the function

$f(t)$, then we are trying to find the *inverse Laplace transform.* So, according to our last example, $\mathcal{L}^{-1}\left(\frac{1}{s+7}\right) = e^{-7t}$.

If you look at the MA 345 course website, you will find a link to an excellent video on how the definition of the Laplace transform can be motivated.

You will have to learn three skills:

a. How to compute a Laplace transform.
b. How to compute an inverse Laplace transform.
c. How to use Laplace transforms to solve differential equation.

It will be a little frustrating to you that you won't get to see how to solve differential equations with Laplace transforms until you are good at calculation both Laplace transforms and their inverses. Be patient - we'll get there. Right now, let's focus on how to calculate a Laplace transform.

We have already seen the Laplace transform of e^{-7t}. If we replace -7 with the constant λ, we obtain the formula:

$$\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$$

Of course, we have to restrict s so that the integral converges. When we looked at $\int_0^\infty e^{-7t}e^{-st}\,dt$, we had to have $s+7$ positive for the integral to converge. In the case of $\int_0^\infty e^{\lambda t}e^{-st}\,dt$, we will need $s-\lambda$ to be positive to get a convergent integral. In most cases, s has to be restricted in some way to guarantee convergence. From now on, we will simply assume that s is suitably restricted whenever we need to do a Laplace transform.

A special case of $\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$ is when $\lambda = 0$

$$\mathcal{L}(1) = \mathcal{L}\left(e^{0t}\right) = \frac{1}{s-0} = \frac{1}{s}$$

The Laplace transform is a *linear* operation because integration is a linear operation. If a and b are constants then:

$$\begin{aligned}
\mathcal{L}(af(t)+bg(t)) &= \int_0^\infty (af(t)+bg(t))e^{-st}\,dt \\
&= a\int_0^\infty f(t)e^{-st}\,dt + b\int_0^\infty g(t)e^{-st}\,dt \\
&= a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))
\end{aligned}$$

This will help use figure out Laplace transforms of different expressions without resorting to integration each time. For example:

$$\mathcal{L}\left(1+e^{2t}+4e^{3t}\right)=\mathcal{L}\left(1\right)+\mathcal{L}\left(e^{2t}\right)+4\mathcal{L}\left(e^{3t}\right)=\frac{1}{s}+\frac{1}{s-2}+\frac{4}{s-3}$$

We will need to be able to calculate Laplace transforms of expressions involving t^n. We already know how to do this when $n=0$ because $\mathcal{L}\left(t^0\right)=\mathcal{L}\left(1\right)=\frac{1}{s}$. As soon as $n>0$, we will need integration by parts:

$$\mathcal{L}\left(t^1\right)=\int_0^\infty te^{-st}\,dt=\lim_{T\to\infty}\frac{-T}{se^{sT}}+\frac{1}{s}\int_0^\infty e^{-st}\,dt=0+\frac{1}{s}\mathcal{L}\left(1\right)=\frac{1}{s^2}$$

$$\mathcal{L}\left(t^2\right)=\int_0^\infty te^{-st}\,dt=\lim_{T\to\infty}\frac{-T^2}{se^{sT}}+\frac{2}{s}\int_0^\infty te^{-st}\,dt=0+\frac{2}{s}\mathcal{L}\left(t\right)=\frac{2}{s^3}$$

To conclude that $\frac{T}{se^{sT}}$ and $\frac{T^2}{se^{sT}}$ approach 0, we use L'Hôpital's Rule. We also need $s>0$ to guarantee convergence. There is a broad category of functions $f(t)$ that have the property that:

$$\lim_{T\to\infty}\frac{f(T)}{e^{sT}}=0$$

These functions are said to be of *exponential order* and include polynomials, $\sin\omega t$, $\cos\omega t$, $e^{\lambda t}$ or any linear combination or product of such functions. These are the only functions we will need to take Laplace transforms of when we solve differential equations. If $\frac{f(T)}{e^{sT}}$ approaches 0, then we can obtain a general formula for the Laplace transform of a derivative $f'(t)$.

$$\begin{aligned}\mathcal{L}\left(f'(t)\right)&=\int_0^\infty f'(t)e^{-st}\,dt\\&=\lim_{T\to\infty}\frac{f(T)}{e^{sT}}-f(0)+s\int_0^\infty f(t)e^{-st}\,dt\quad\text{(integration by parts)}\\&=0-f(0)+s\int_0^\infty f(t)e^{-st}\,dt\\&=s\int_0^\infty f(t)e^{-st}\,dt-f(0)\end{aligned}$$

In other words:

$$\mathcal{L}\left(f'(t)\right) = s\mathcal{L}\left(f(t)\right) - f(0)$$

This is an extremely useful formula for obtaining Laplace transforms. For example, if $f(t) = t^n$ then $f'(t) = nt^{n-1}$ so:

$$\mathcal{L}\left(nt^{n-1}\right) = s\mathcal{L}\left(t^n\right) - 0^n = s\mathcal{L}\left(t^n\right) \quad \text{(assuming } n > 0\text{)}$$

Linearity implies that $n\mathcal{L}\left(t^{n-1}\right) = s\mathcal{L}\left(t^n\right)$ so we get a reduction formula:

$$\mathcal{L}\left(t^n\right) = \frac{n}{s}\mathcal{L}\left(t^{n-1}\right)$$

We can now get a general formula for $\mathcal{L}\left(t^n\right)$

$$\begin{aligned}
\mathcal{L}\left(t^1\right) &= \frac{1}{s}\mathcal{L}\left(t^0\right) = \frac{1}{s}\mathcal{L}\left(1\right) = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2} \\
\mathcal{L}\left(t^2\right) &= \frac{2}{s}\mathcal{L}\left(t^1\right) = \frac{2}{s}\mathcal{L}\left(t\right) = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3} \\
\mathcal{L}\left(t^3\right) &= \frac{3}{s}\mathcal{L}\left(t^2\right) = \frac{3}{s}\mathcal{L}\left(t^2\right) = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{(3)(2)}{s^4} \\
\mathcal{L}\left(t^4\right) &= \frac{4}{s}\mathcal{L}\left(t^3\right) = \frac{4}{s}\mathcal{L}\left(t^3\right) = \frac{4}{s}\cdot\frac{(3)(2)}{s^4} = \frac{(4)(3)(2)}{s^5}
\end{aligned}$$

The general formula is:

$$\mathcal{L}\left(t^n\right) = \frac{n!}{s^{n+1}}$$

We can now use linearity to get the Laplace transform of any polynomial. For example:

$$\mathcal{L}\left(2t^2 - 3t + 4\right) = 2\mathcal{L}\left(t^2\right) - 3\mathcal{L}\left(t\right) + 4\mathcal{L}\left(1\right) = 2\cdot\frac{2}{s^3} - 3\cdot\frac{1}{s^2} + 4\cdot\frac{1}{s} = \frac{4}{s^3} - \frac{3}{s^2} + \frac{4}{s}$$

The formula $\mathcal{L}\left(f'(t)\right) = s\mathcal{L}\left(f(t)\right) - f(0)$ is an algebraic relationship between the Laplace transform of a function and the Laplace transform of its derivative. This is actually the reason that Laplace transforms are useful in solving differential equations. If we take the Laplace transform of both sides of a differential equation, we will obtain an algebraic equation involving the

Laplace transform of the solution $y(t)$. Algebraic equations are usually easier to solve than differential equations. More details on this later on when we are finally ready to solve differential equations using Laplace transforms.

We have obtained formulas for the Laplace transforms of $e^{\lambda t}$ and t^n. It is not hard to put these together and get a formula for $\mathcal{L}\left(t^n e^{\lambda t}\right)$. We already know that:

$$\mathcal{L}\left(t^n\right) = \int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}$$

Now, compare this with:

$$\mathcal{L}\left(t^n e^{\lambda t}\right) = \int_0^\infty t^n e^{\lambda t} e^{-st}\, dt = \int_0^\infty t^n e^{-(s-\lambda)t}\, dt$$

The only difference between the two integrals is the the constant s in the exponent of the first integral is replace by $s - \lambda$ in the second integral. Therefore,

$$\mathcal{L}\left(t^n e^{\lambda t}\right) = \int_0^\infty t^n e^{\lambda t} e^{-st}\, dt = \int_0^\infty t^n e^{-(s-\lambda)t}\, dt = \frac{n!}{(s-\lambda)^n + 1}$$

To get the Laplace transforms of sine and cosine, we begin with Euler's formulas:

$$\begin{aligned} e^{i\omega t} &= \cos\omega t + i\sin\omega t \\ e^{-i\omega t} &= \cos\omega t - i\sin\omega t \end{aligned}$$

If we add these equations, we get:

$$e^{i\omega t} + e^{-i\omega t} = 2\cos\omega t \quad \text{so} \quad \cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

If we subtract the equations instead of adding them, we get:

$$e^{i\omega t} - e^{-i\omega t} = 2i\sin\omega t \quad \text{so} \quad \sin\omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

These are sometimes referred to as the *backwards* Euler formulas.

Now, if we use the linearity property of Laplace transforms, combined with the formula $\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$, we get:

$$\mathcal{L}(\sin \omega t) = \frac{1}{2i}\left(\mathcal{L}\left(e^{i\omega t}\right) - \mathcal{L}\left(e^{-i\omega t}\right)\right) = \frac{1}{2i}\left(\frac{1}{s-i\omega t} - \frac{1}{s+i\omega t}\right)$$

$$\mathcal{L}(\cos \omega t) = \frac{1}{2}\left(\mathcal{L}\left(e^{i\omega t}\right) + \mathcal{L}\left(e^{-i\omega t}\right)\right) = \frac{1}{2}\left(\frac{1}{s-i\omega t} + \frac{1}{s+i\omega t}\right)$$

Combine these fractions. Note that $(s-i\omega)(s+i\omega) = s^2 - i^2\omega^2 = s^2 + \omega^2$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

Just as we generalized from $\mathcal{L}(t^n)$ to $\mathcal{L}\left(t^n e^{\lambda t}\right)$, we can do the same thing for sine and cosine.

$$\mathcal{L}\left(e^{\lambda t}\sin \omega t\right) = \frac{\omega}{(s-\lambda)^2+\omega^2} \qquad \mathcal{L}\left(e^{\lambda t}\cos \omega t\right) = \frac{s-\lambda}{(s-\lambda)^2+\omega^2}$$

We have now built up enough formulas to summarize in a table:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left(e^{\lambda t}t^n\right) = \frac{n!}{(s-\lambda)^{n+1}}$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\left(e^{\lambda t}\sin \omega t\right) = \frac{\omega}{(s-\lambda)^2+\omega^2}$$

$$\mathcal{L}\left(e^{\lambda t}\cos \omega t\right) = \frac{s-\lambda}{(s-\lambda)^2+\omega^2}$$

It will be convenient to refer to this table whenever we need the Laplace transform of a function rather than to go through the integration argument all over again.

There is one final case to consider where, at least for now, we will need to refer to the integral definition of Laplace transform. Suppose $f(t)$ is a discontinuous function defined by:

$$f(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t > 1 \end{cases}$$

The calculation of $\mathcal{L}\left(f(t)\right)$ requires splitting the integral up into 2 different integrals:

$$\mathcal{L}\left(f(t)\right) = \int_0^1 f(t)e^{-st}\,dt + \int_1^\infty f(t)e^{-st}\,dt$$

We now use the appropriate formula for $f(t)$ for each of these integrals:

$$\mathcal{L}\left(f(t)\right) = \int_0^1 te^{-st}\,dt + \int_1^\infty 0 \cdot e^{-st}\,dt$$

The second integral equals 0. The first integral can be done using integration by parts.

$$\mathcal{L}\left(f(t)\right) = \frac{1}{s^2} - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}$$

There is an elegant way to deal with Laplace transforms of discontinous functions, but we'll hold off on that until we get to *unit step functions*.