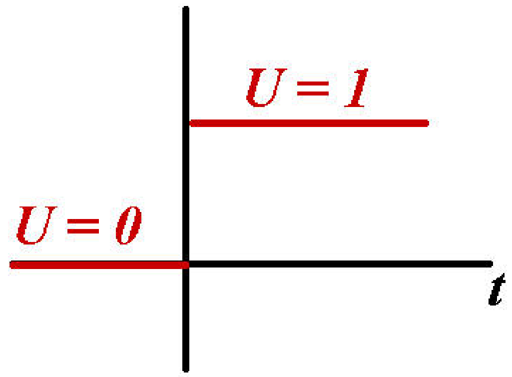
Laplace Transform of the Unit Step Function
Jacobs

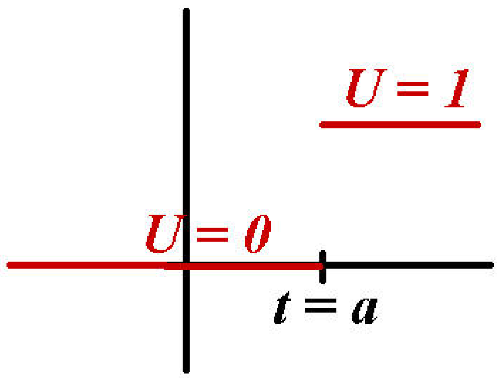
One of the advantages of using Laplace transforms to solve differential equations is the way it simplifies problems involving functions that undergo sudden jumps. Consider the function $\mathcal{U}(t)$ defined as:

$$\mathcal{U}(t) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

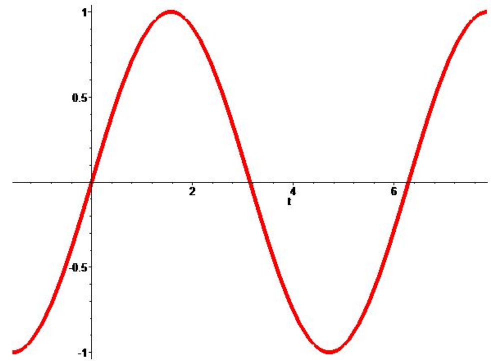
This function is called the *unit step function*. Some texts refer to this as the *Heaviside* step function. Here's the graph of $\mathcal{U}(t)$



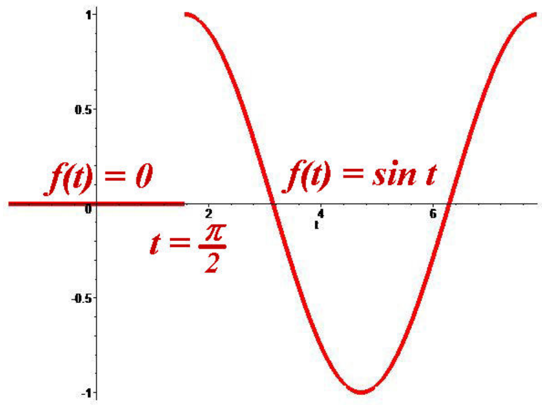
The graph of the function $y = \mathcal{U}(t-a)$ is similar except that it is 0 for $t < a$ and it's 1 for $t \geq a$.



We can use the unit step function to select a portion of a graph to plot. For example, the plot of $y = \sin t$ is:

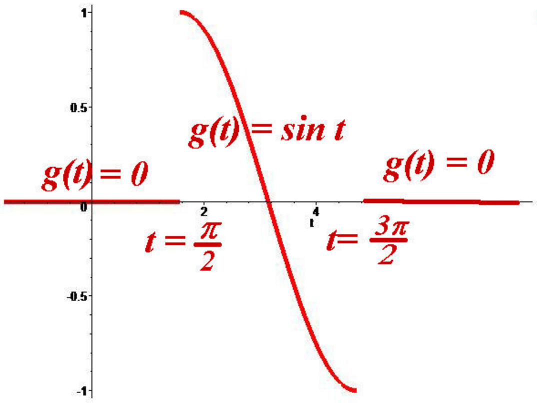


Now, compare this graph to the graph of $f(t) = \sin t \cdot \mathcal{U}\left(t - \frac{\pi}{2}\right)$



The function $f(t)$ is 0 for $t < \frac{\pi}{2}$ and it's $\sin t$ for $t \geq \frac{\pi}{2}$. Now, compare the graph of $f(t)$ with the graph of $g(t) = \sin t \cdot \left(\mathcal{U}\left(t - \frac{\pi}{2}\right) - \mathcal{U}\left(t - \frac{3\pi}{2}\right)\right)$

$$g(t) = (\sin t) \cdot \left(\mathcal{U}\left(t - \frac{\pi}{2}\right) - \mathcal{U}\left(t - \frac{3\pi}{2}\right) \right) = \begin{cases} 0 & \text{for } -\infty < t < \frac{\pi}{2} \\ \sin t & \text{for } \frac{\pi}{2} \le t \le \frac{3\pi}{2} \\ 0 & \text{for } \frac{3\pi}{2} < t < \infty \end{cases}$$



We are going to need to calculate Laplace transforms of expressions of the form $g(t)\mathcal{U}(t-a)$ in order to use Laplace transforms to solve problems $P(D)y = g(t)$ involving abrupt changes in the function $g(t)$.

Before we can do that, take a look at the following integrals:

$$\int_0^2 t^2\, dt = \frac{8}{3} \qquad \int_0^2 u^2\, du = \frac{8}{3} \qquad \int_0^2 v^2\, dv = \frac{8}{3}$$

The name of the variable of integration doesn't affect the answer, so it is sometimes referred to as a *dummy variable*. The same thing happens with Laplace transforms. We know, for example, that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$. In terms of integrals, $\int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}$. However, the following integrals have exactly the same final value:

$$\int_0^\infty u^n e^{-su}\, du = \frac{n!}{s^{n+1}} \qquad \int_0^\infty v^n e^{-sv}\, dv = \frac{n!}{s^{n+1}}$$

Therefore, whether we write $\mathcal{L}(f(t))$ or $\mathcal{L}(f(u))$ or $\mathcal{L}(f(v))$ we still get the same function of s as a result.

$$\mathcal{L}(f(t)) = \mathcal{L}(f(u)) = \mathcal{L}(f(v))$$

With that in mind, let's calculate $\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right)$

$$\begin{aligned}\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) &= \int_0^\infty g(t)\mathcal{U}(t-a)e^{-st}\,dt \\ &= \int_0^a g(t)\cdot 0\cdot e^{-st}\,dt + \int_a^\infty g(t)\cdot 1\cdot e^{-st}\,dt \\ &= \int_a^\infty g(t)e^{-st}\,dt\end{aligned}$$

Now, make a change of variable $u = t - a$

$$\begin{aligned}\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) &= \int_a^\infty g(t)e^{-st}\,dt \\ &= \int_0^\infty g(u+a)e^{-s(u+a)}\,du \\ &= e^{-as}\int_0^\infty g(u+a)e^{-su}\,du \\ &= e^{-as}\mathcal{L}\left(g(u+a)\right) \\ &= e^{-as}\mathcal{L}\left(g(t+a)\right)\end{aligned}$$

The same change of variable $u = t + a$ gives us a similar formula for $\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right)$

$$\begin{aligned}\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) &= \int_a^\infty g(t-a)e^{-st}\,dt \\ &= \int_0^\infty g(u)e^{-s(u+a)}\,du \\ &= e^{-as}\int_0^\infty g(u)e^{-su}\,du \\ &= e^{-as}\mathcal{L}\left(g(u)\right) \\ &= e^{-as}\mathcal{L}\left(g(t)\right)\end{aligned}$$

We now have the following two formulas for taking Laplace transforms involving the unit step function $\mathcal{U}$

$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t+a)\right)$$

$$\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)$$

Example:
If an electric circuit contains a resistor of resistance R and a capacitor of capacitance C then if $y = y(t)$ is the amount of charge (in coulombs) on the capacitor at time t then Kirchoff's Law implies that y satisfies the following differential equation:

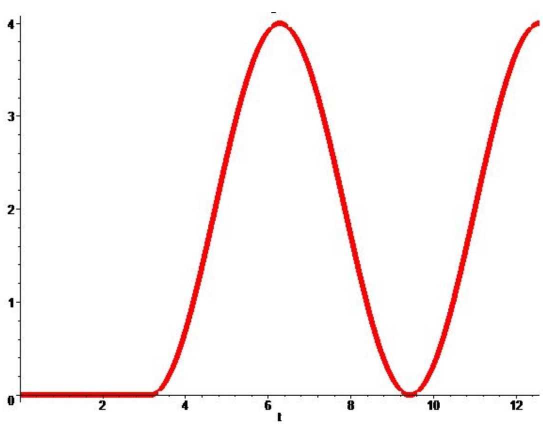
$$R\frac{dy}{dt} + \frac{1}{C}y = \mathcal{E}(t)$$

where $\mathcal{E}(t)$ is a time-dependent electromotive force (voltage) applied to the circuit at time t. Let's take a simple example where $R = 1$ ohm and $C = 1$ farad. For simplicity, take the initial condition $y(0) = 0$.

$$y' + y = \mathcal{E}(t) \qquad \text{where } y(0) = 0$$

Suppose that the voltage is $\mathcal{E}(t) = 0$ for the first π seconds and then a switch is flipped and $\mathcal{E}(t) = 2 + 2\cos t$ for $t \geq \pi$.

$$\mathcal{E}(t) = \begin{cases} 0 & \text{for } 0 \leq t < \pi \\ 2 + 2\cos t & \text{for } t \geq \pi \end{cases}$$



Notice that $\mathcal{E}(t) = (2 + 2\cos t)\mathcal{U}(t - \pi)$. Let's solve the differential equation and obtain a formula for $y(t)$ (how charge on the capacitor varies with time).

Take the Laplace transform of both sides of the equation:

$$y' + y = (2 + 2\cos t)\mathcal{U}(t - \pi) \qquad \text{where } y(0) = 0$$

$$\mathcal{L}\left(y'\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left((2 + 2\cos t)\mathcal{U}(t - \pi)\right)$$

We replace $\mathcal{L}\left(y'\right)$ with $s\mathcal{L}\left(y\right) - y(0) = s\mathcal{L}\left(y\right) - 0 = s\mathcal{L}\left(y\right)$.

$$s\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left((2 + 2\cos t)\mathcal{U}(t - \pi)\right)$$

Now, we use the formula $\mathcal{L}\left(g(t)\mathcal{U}(t - a)\right) = e^{-as}\mathcal{L}\left(g(t + a)\right)$
with $g(t) = 2 + 2\cos t$ and $a = \pi$

$$s\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = e^{-\pi s}\mathcal{L}\left((2 + 2\cos(t + \pi)\right)$$

$$(s + 1)\mathcal{L}\left(y\right) = e^{-\pi s}\mathcal{L}\left((2 - 2\cos t)\right)$$

$$(s + 1)\mathcal{L}\left(y\right) = e^{-\pi s}\left(\frac{2}{s} - \frac{2s}{s^2 + 1}\right) = e^{-\pi s}\left(\frac{2}{s(s^2 + 1)}\right)$$

$$\mathcal{L}\left(y\right) = e^{-\pi s}\left(\frac{2}{s(s + 1)(s^2 + 1)}\right)$$

We will have the answer to our differential equation if we can only find the inverse Laplace transform. As usual, we can simplify the problem by using a partial fraction decomposition.

$$\mathcal{L}\left(y\right) = e^{-\pi s}\left(\frac{2}{s} - \frac{1}{s + 1} - \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}\right)$$

$$\mathcal{L}\left(y\right) = e^{-\pi s}\mathcal{L}\left(2 - e^{-t} - \cos t - \sin t\right)$$

We now use the formula $\mathcal{L}\left(g(t - a)\mathcal{U}(t - a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)$ with $a = \pi$ and $g(t) = 2 - e^{-t} - \cos t - \sin t$

$$\mathcal{L}\left(y\right) = \mathcal{L}\left(g(t - \pi)\mathcal{U}(t - \pi)\right)$$

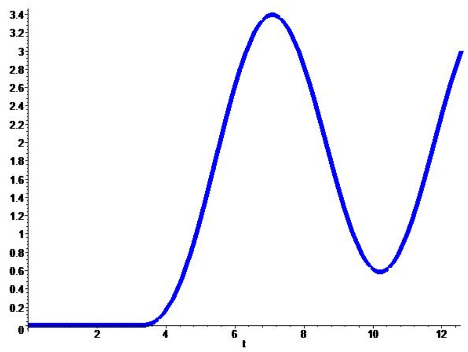
Now, take the inverse Laplace transform:

$$y(t) = g(t - \pi)\mathcal{U}(t - \pi) = \left(2 - e^{-(t - \pi)} - \cos(t - \pi) - \sin(t - \pi)\right)\mathcal{U}(t - \pi)$$

We can simplify using the trigonometric identities $\cos(t-\pi) = -\cos t$ and $\sin(t-\pi) = -\sin t$

$$y(t) = \mathcal{U}(t-\pi)\left(2 - e^{\pi - t} + \cos t + \sin t\right)$$

What this means is that $y(t) = 2 - e^{\pi-t} + \cos t + \sin t$ for $t \geq \pi$ but $y(t) = 0$ for $t < \pi$



Example:
Solve the following differential equation:

$$y'' + y = 1 + \mathcal{U}(t-2\pi) \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 0$$

Such an equation would arise if we had a circuit with an inductor of $L = 1$ henry, a resistance of $R = 0$ (in other words, no resistor) and a capacitor of $C = 1$ farad. The voltage source $\mathcal{E}(t)$ is 1 volt for $0 \leq t < 2\pi$ and 2 volts after that.

We start, as always by taking $\mathcal{L}$ of both sides.

$$\mathcal{L}\left(y''\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(1\right) + \mathcal{L}\left(\mathcal{U}(t-2\pi)\right)$$

$$s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) + \mathcal{L}\left(y\right) = \frac{1}{s} + \mathcal{L}\left(\mathcal{U}(t-2\pi)\right)$$

Since $y(0) = 0$ and $y'(0) = 0$ this simplifies to:

$$\left(s^2+1\right)\mathcal{L}\left(y\right) = \frac{1}{s} + \mathcal{L}\left(\mathcal{U}(t-2\pi)\right)$$

To get $\mathcal{L}(\mathcal{U}(t-2\pi))$, we use $\mathcal{L}(g(t)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t+a))$ with $g(t) = 1$ and $a = 2\pi$.

$$\left(s^2+1\right)\mathcal{L}(y) = \frac{1}{s} + e^{-2\pi s}\frac{1}{s}$$

Now, we solve for $\mathcal{L}(y)$ and try to find the inverse Laplace transform.

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{s(s^2+1)} + e^{-2\pi s}\left(\frac{1}{s(s^2+1)}\right)\\ &= \frac{1}{s} - \frac{s}{s^2+1} + e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\\ &= \mathcal{L}(1-\cos t) + e^{-2\pi s}\mathcal{L}(1-\cos t)\end{aligned}$$

Now, we can use the formula $\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t))$ with $a = 2\pi$ and $g(t) = 1-\cos t$

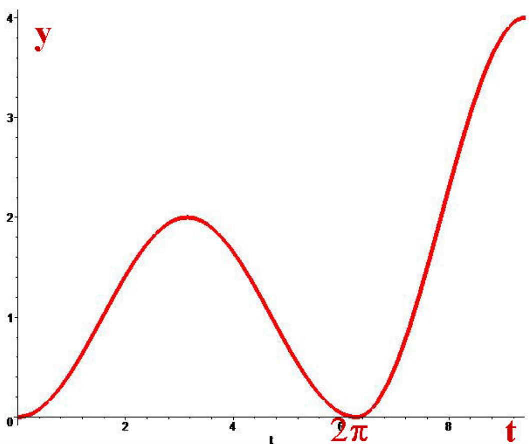
$$\mathcal{L}(y) = \mathcal{L}(1-\cos t) + \mathcal{L}((1-\cos(t-2\pi))\mathcal{U}(t-2\pi))$$

$$y = 1-\cos t + (1-\cos(t-2\pi))\mathcal{U}(t-2\pi)$$

Since $\cos(t-2\pi) = \cos t$, we are left with:

$$y = 1-\cos t + (1-\cos t)\mathcal{U}(t-2\pi)$$

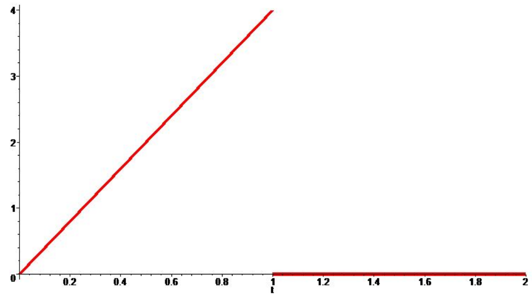
Here's a plot of the solution. You can see the change in the solution starting from $t = 2\pi$, when the voltage is increased.



Example:
Solve the following differential equation:

$$y'' + 4y' + 4y = 4t(1 - \mathcal{U}(t-1)) \qquad \text{where } y(0) = y'(0) = 0$$

If we want to look at this as an electric circuit problem, then the voltage $\mathcal{E}(t) = 4t(1 - \mathcal{U}(t-1))$ rises steadily until $t = 1$ and then cuts off.



We start by taking Laplace transform of both sides:

$$\mathcal{L}(y'') + 4\mathcal{L}(y') + 4\mathcal{L}(y) = \mathcal{L}(4t(1 - \mathcal{U}(t-1))$$

Since $y(0) = y'(0) = 0$, the left hand side simplifies quickly

$$(s^2 + 4s + 4)\mathcal{L}(y) = \mathcal{L}(4t) - \mathcal{L}(4t\mathcal{U}(t-1))$$

$$(s+2)^2\mathcal{L}(y) = \frac{4}{s^2} - e^{-1s}\mathcal{L}(4(t+1))$$

Notice that we just used the formula $\mathcal{L}(g(t)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t+a))$ where $a = 1$ and $g(t) = 4t$

$$(s+2)^2\mathcal{L}(y) = \frac{4}{s^2} - e^{-1s}\left(\frac{4}{s^2} + \frac{4}{s}\right)$$

Combining fractions and solving for $\mathcal{L}(y)$

$$\mathcal{L}(y) = \frac{4}{s^2(s+2)^2} - e^{-1s}\left(\frac{4s+4}{s^2(s+2)^2}\right)$$

Here is where we use partial fractions decomposition if we want to get the inverse Laplace transform.

$$\mathcal{L}\left(y\right)=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{(s+2)^2}+\frac{1}{s+2}-e^{-1s}\left(\frac{1}{s^2}-\frac{1}{(s+2)^2}\right)$$

$$\mathcal{L}\left(y\right)=\mathcal{L}\left(t-1+te^{-2t}+e^{-2t}\right)-e^{-1s}\mathcal{L}\left(t-te^{-2t}\right)$$

We can make use of the formula $\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right)=e^{-as}\mathcal{L}\left(g(t)\right)$ with $a=1$ and $g(t)=t-te^{-2t}$.
Note that $g(t-1)=t-1-(t-1)e^{-2(t-1}=t-1-(t-1)e^{2-2t}$

We can now take the inverse Laplace transform and get the solution.

$$y(t)=t-1+te^{-2t}+e^{-2t}-(t-1)\mathcal{U}(t-1)\cdot\left(1-e^{2-2t}\right)$$

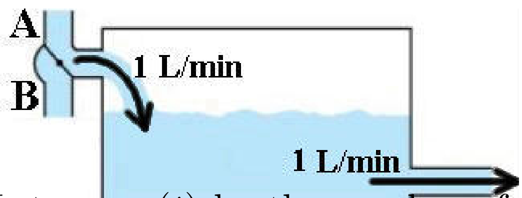
Example - The Valve Problem

Fluid is flowing into container at 1 liter/min
Fluid is flowing out at 1 liter/min.
For simplicity, assume the container has 1 liter of fluid at all times.

Initially, the tank contains nothing but pure water.

For $t<2$, the fluid coming in has 1 gram of salt per liter.

For $t\geq 2$, the fluid coming in has only pure water.



Let $y=y(t)$ be the number of grams of salt in the tank after t minutes. Let's set up the differential equation that correctly determines y

$$\text{Rate Out}=\frac{1\text{ liter}}{1\text{ min}}\cdot\frac{y\text{ grams}}{1\text{ liter}}=y\,\frac{\text{grams}}{\text{min}}$$

$$\text{Rate In}=\begin{cases}1\text{ gram/min} & \text{for } t\leq 2\\ 0 & \text{for } t>2\end{cases}=1-\mathcal{U}(t-2)$$

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out} = 1 - \mathcal{U}(t-2) - y$$

$$y' + y = 1 - \mathcal{U}(t-2)$$

$$\mathcal{L}(y') + \mathcal{L}(y) = \frac{1}{s} - e^{-2s} \cdot \frac{1}{s}$$

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{s(s+1)} - e^{-2s} \cdot \frac{1}{s(s+1)} \\ &= \frac{1}{s} - \frac{1}{s+1} - e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right) \\ &= \mathcal{L}\left(1 - e^{-t}\right) - e^{-2s} \cdot \mathcal{L}\left(1 - e^{-t}\right)\end{aligned}$$

$$y(t) = 1 - e^{-t} - \mathcal{U}(t-2)\left(1 - e^{2-t}\right)$$