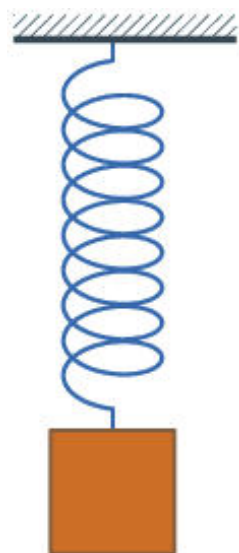


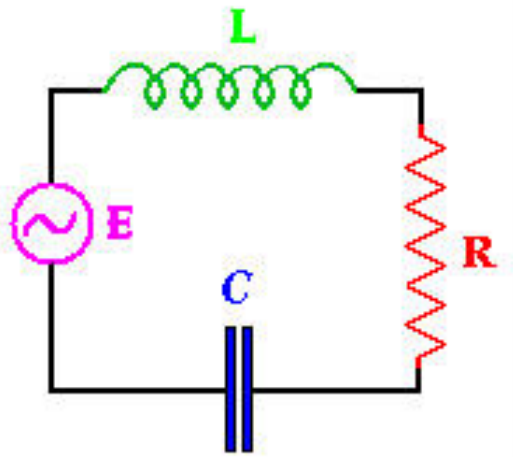
**Laplace Transform
of the Unit Step Function
Jacobs**

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = f(t)$$

$$m \frac{d^2 y}{dt^2} + \beta \frac{dy}{dt} + ky = f(t)$$

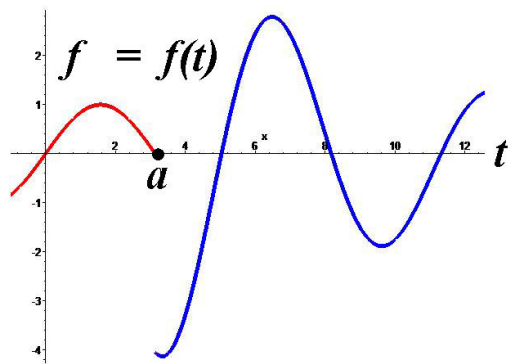


$$L \frac{d^2 y}{dt^2} + R \frac{dy}{dt} + \frac{1}{C} y = \mathcal{E}(t)$$



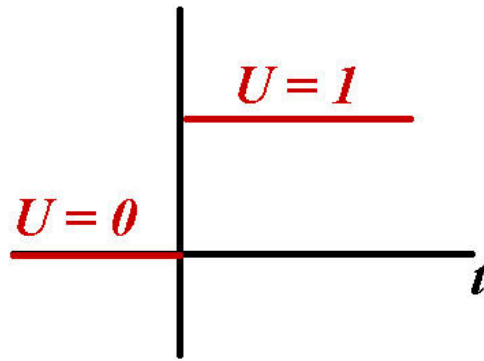
Solve $a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + y = f(t)$

when $f(t)$ is a discontinuous function



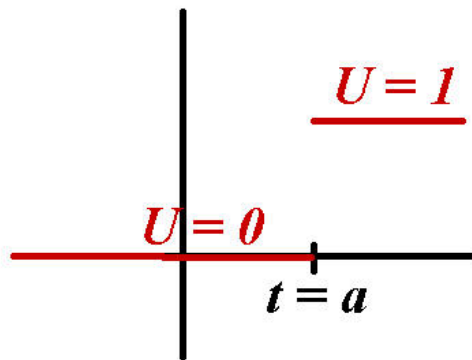
The Unit Step Function

$$\mathcal{U}(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

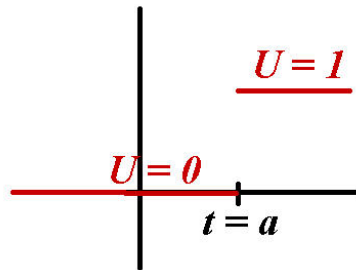


The Shifted Unit Step Function

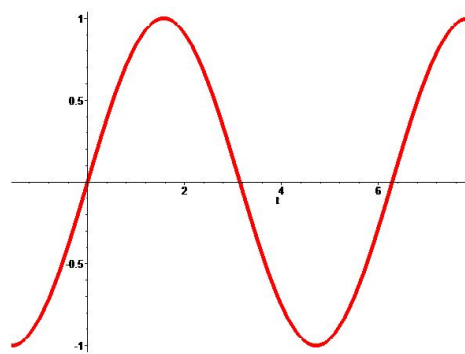
$$\mathcal{U}(t - a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$



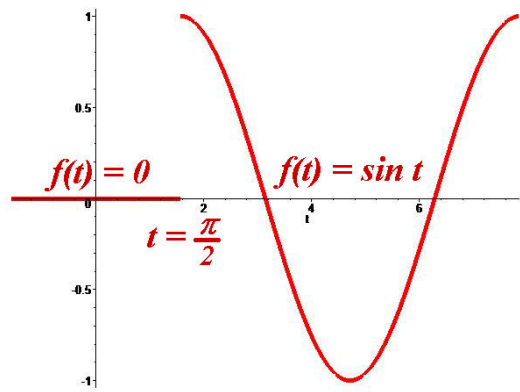
$$\begin{aligned}
 \mathcal{L}(\mathcal{U}(t-a)) &= \int_0^{\infty} \mathcal{U}(t-a)e^{-st} dt \\
 &= \int_a^{\infty} 1 \cdot e^{-st} dt \\
 &= e^{-as} \cdot \frac{1}{s}
 \end{aligned}$$



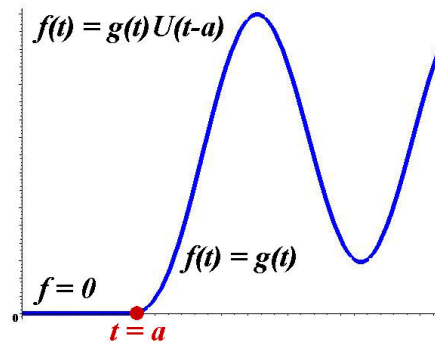
$$y(t) = \sin t$$



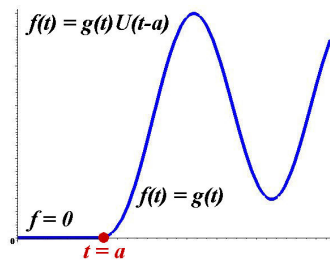
$$f(t) = \sin t \cdot \mathcal{U}\left(t - \frac{\pi}{2}\right)$$



$$f(t) = g(t)\mathcal{U}(t - a)$$



$$\begin{aligned}
\mathcal{L}(g(t)\mathcal{U}(t-a)) &= \int_0^\infty g(t)\mathcal{U}(t-a)e^{-st} dt \\
&= \int_a^\infty g(t)e^{-st} dt \\
&= \int_0^\infty g(x+a)e^{-s(x+a)} dx \quad \text{where } x = t - a \\
&= e^{-sa} \int_0^\infty g(x+a)e^{-sx} dx
\end{aligned}$$



$$\int_0^\infty g(t)\mathcal{U}(t-a)e^{-st}dt=e^{-sa}\int_0^\infty g(x+a)e^{-sx}dx$$

$$\int_0^\infty g(t)\mathcal{U}(t-a)e^{-st}dt=e^{-sa}\int_0^\infty g(t+a)e^{-st}dt$$

$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right)=e^{-sa}\mathcal{L}\left(g(t+a)\right)$$

$$\mathcal{L} (g(t)\mathcal{U}(t-a)) = e^{-as} \mathcal{L} (g(t+a))$$

$$\mathcal{L} (g(t-a)\mathcal{U}(t-a)) = e^{-as} \mathcal{L} (g(t))$$

Useful for taking Laplace transform of (function)(unit step)

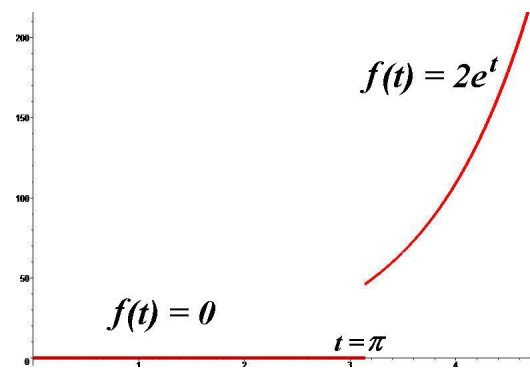
$$\mathcal{L}(g(t)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t+a))$$

Useful for taking inverse Laplace transforms of $e^{-as}G(s)$ where $G(s) = \mathcal{L}(g(t))$

$$\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t))$$

$$f(t) = 2e^t \mathcal{U}(t - \pi)$$

Find the Laplace transform of $f(t)$



$$f(t) = 2e^t \mathcal{U}(t - \pi)$$

Find the Laplace transform of $f(t)$

$$\mathcal{L}(g(t)\mathcal{U}(t - a)) = e^{-as} \mathcal{L}(g(t + a))$$

$$\mathcal{L}(2e^t \mathcal{U}(t - \pi)) = e^{-\pi s} \mathcal{L}(g(t + \pi))$$

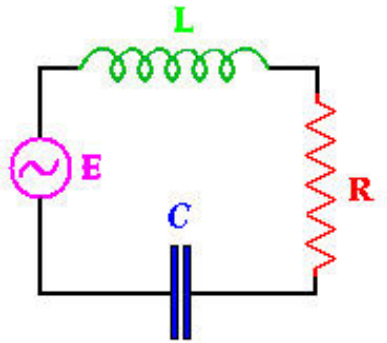
$$f(t) = 2e^t \mathcal{U}(t - \pi)$$

Find the Laplace transform of $f(t)$

$$\mathcal{L}(g(t)\mathcal{U}(t - a)) = e^{-as} \mathcal{L}(g(t + a))$$

$$\begin{aligned} \mathcal{L}(2e^t \mathcal{U}(t - \pi)) &= e^{-\pi s} \mathcal{L}(g(t + \pi)) \\ &= e^{-\pi s} \mathcal{L}(2e^{t+\pi}) \\ &= 2e^\pi e^{-\pi s} \mathcal{L}(e^t) \\ &= 2e^\pi \cdot \frac{e^{-\pi s}}{s - 1} \end{aligned}$$

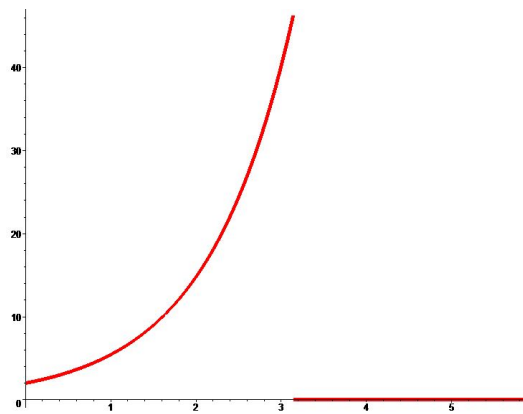
$$L \frac{d^2 y}{dt^2} + R \frac{dy}{dt} + \frac{1}{C} y = \mathcal{E}(t)$$



Take $L = 1$, $R = 0$ and $C = 1$

$$y'' + y = \mathcal{E}(t)$$

$$\mathcal{E}(t) = 2e^t(1 - \mathcal{U}(t - \pi))$$



$$y'' + y = 2e^t(1 - \mathcal{U}(t - \pi)) \quad \text{where } y(0) = y'(0) = 0$$

$$y'' + y = 2e^t - 2e^t\mathcal{U}(t - \pi)$$

$$y'' + y = 2e^t(1 - \mathcal{U}(t - \pi)) \quad \text{where } y(0) = y'(0) = 0$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(2e^t) - \mathcal{L}(2e^t\mathcal{U}(t - \pi))$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{2}{s-1} - 2e^\pi \cdot \frac{e^{-\pi s}}{s-1}$$

$$s^2\mathcal{L}(y) + \mathcal{L}(y) = \frac{2}{s-1} - 2e^\pi e^{-\pi s} \cdot \frac{1}{s-1}$$

$$y'' + y = 2e^t(1 - \mathcal{U}(t - \pi)) \quad \text{where } y(0) = y'(0) = 0$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(2e^t) - \mathcal{L}(2e^t\mathcal{U}(t - \pi))$$

$$(s^2 + 1)\mathcal{L}(y) = \frac{2}{s - 1} - 2e^\pi e^{-\pi s} \cdot \frac{1}{s - 1}$$

$$\mathcal{L}(y) = \frac{2}{(s - 1)(s^2 + 1)} - e^\pi e^{-\pi s} \cdot \frac{2}{(s - 1)(s^2 + 1)}$$

$$\mathcal{L}(y) = \frac{2}{(s-1)(s^2+1)} - e^\pi e^{-\pi s} \cdot \frac{2}{(s-1)(s^2+1)}$$

$$\mathcal{L}(y) = \frac{1}{s-1} - \frac{(s+1)}{s^2+1} - e^{\pi-\pi s} \left(\frac{1}{s-1} - \frac{(s+1)}{s^2+1} \right)$$

$$\begin{aligned}
\frac{1}{s-1} - \frac{(s+1)}{s^2+1} &= \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1} \\
&= \mathcal{L}(e^t) - \mathcal{L}(\cos t) - \mathcal{L}(\sin t) \\
&= \mathcal{L}(e^t - \cos t - \sin t)
\end{aligned}$$

Let $g(t) = e^t - \cos t - \sin t$

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{s-1} - \frac{(s+1)}{s^2+1} - e^{\pi-\pi s} \left[\frac{1}{s-1} - \frac{(s+1)}{s^2+1} \right] \\ &= \mathcal{L}(g(t)) - e^{\pi} e^{-\pi s} \mathcal{L}(g(t))\end{aligned}$$

where $g(t) = e^t - \cos t - \sin t$

Now use the formula:

$$\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-as} \mathcal{L}(g(t))$$

$$\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t))$$

If $a = \pi$ and $g(t) = e^t - \cos t - \sin t$

$$\begin{aligned} g(t-\pi) &= e^{t-\pi} - \cos(t-\pi) - \sin(t-\pi) \\ &= e^{t-\pi} + \cos t + \sin t \end{aligned}$$

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{s-1} - \frac{(s+1)}{s^2+1} - e^{\pi-\pi s} \left[\frac{1}{s-1} - \frac{(s+1)}{s^2+1} \right] \\ &= \mathcal{L}(g(t)) - e^{\pi} e^{-\pi s} \mathcal{L}(g(t))\end{aligned}$$

where $g(t) = e^t - \cos t - \sin t$

Now use the formula:

$$\mathcal{L}(g(t-\pi)\mathcal{U}(t-\pi)) = e^{-\pi s} \mathcal{L}(g(t))$$

$$\mathcal{L}\left((e^{t-\pi} + \cos t + \sin t)\mathcal{U}(t-\pi)\right) = e^{-\pi s} \mathcal{L}(g(t))$$

$$g(t) = e^t - \cos t - \sin t$$

$$\begin{aligned}\mathcal{L}(y) &= \mathcal{L}(g(t)) - e^\pi \mathcal{L}(g(t - \pi)\mathcal{U}(t - \pi)) \\ &= \mathcal{L}(g(t) - e^\pi g(t - \pi)\mathcal{U}(t - \pi))\end{aligned}$$

$$g(t) = e^t - \cos t - \sin t$$

$$\begin{aligned}\mathcal{L}(y) &= \mathcal{L}(g(t)) - e^\pi \mathcal{L}(g(t - \pi)\mathcal{U}(t - \pi)) \\ &= \mathcal{L}(g(t) - e^\pi g(t - \pi)\mathcal{U}(t - \pi))\end{aligned}$$

$$\begin{aligned}y &= g(t) - e^\pi g(t - \pi)\mathcal{U}(t - \pi) \\ &= e^t - \cos t - \sin t - e^\pi [e^{t-\pi} + \cos t + \sin t] \mathcal{U}(t - \pi) \\ &= e^t - \cos t - \sin t - [e^t + e^\pi(\cos t + \sin t)] \mathcal{U}(t - \pi)\end{aligned}$$

$$y = e^t - \cos t - \sin t - \left[e^t + e^\pi (\cos t + \sin t) \right] \mathcal{U}(t - \pi)$$

