$$4\langle 2,\, 3\rangle = \langle 8,\, 12\rangle$$

$$0\langle x_1,\, x_2\rangle = \langle 0x_1,\, 0x_2\rangle = \langle 0,\, 0\rangle$$

$$0\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

$$(-3)\begin{pmatrix} 5 \\ 10 \\ 2 \end{pmatrix} = \begin{pmatrix} -15 \\ -30 \\ -6 \end{pmatrix}$$

$$2\begin{pmatrix} 1 & 2 & 4 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = ?$$

$$\begin{pmatrix} 2 & 4 & 8 \\ 10 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

$c\mathbf{A}$

Multiplication of a matrix by a number:

$$c\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}$$

$$= \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & ca_{23} & \cdots & ca_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ ca_{n1} & ca_{n2} & ca_{n3} & \cdots & ca_{nm} \end{pmatrix}$$

$$\langle 2,\ 3,\ 1\rangle + \langle 1,\ 4,\ 0\rangle = \langle 3,\ 7,\ 1\rangle$$

$$\begin{pmatrix} 10 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ -1 & 1 & -2 \end{pmatrix} = ?$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 & 4 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = ?$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{A} + \mathbf{O} = \mathbf{A}$$

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = ?$$

UNDEFINED

$\mathbf{A} + \mathbf{B}$ is defined only when:

number of rows of $\mathbf{A}$ = number of rows of $\mathbf{B}$

number of columns of $\mathbf{A}$ = number of columns of $\mathbf{B}$

Matrix Multiplication

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{AX} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\text{Let } \mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

AB is defined by the following formula:

$$\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Example:

Multiply the following two matrices:

$$\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 8 \\ 2 & 7 \end{pmatrix}$$

Multiply the following two matrices:

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 12 & 2 \end{pmatrix}$$

Note that $\mathbf{AB}$ is not necessarily the same as $\mathbf{BA}$

Matrix Multiplication

To obtain the entry in the i^{th} row, j^{th} column of **AB**, take the dot product of the i^{th} row of **A** with the j^{th} column of **B**.

$$
\begin{matrix} & j^{th} column \\ i^{th} row & \begin{pmatrix} & \vdots & \\ & \vdots & \\ \dots & \sum_{k=1}^{n} a_{ik}b_{kj} & \dots \\ & \vdots & \\ & \vdots & \end{pmatrix} = \end{matrix}
$$

$$
\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1q} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{nq} \end{pmatrix}
$$

Example:

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 14 \\ -1 & -1 \\ 5 & 0 \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

An n by n identity matrix is a matrix of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

An identity matrix **I** has the property that

$$\mathbf{IA} = \mathbf{A} \text{ and } \mathbf{AI} = \mathbf{A}$$

for any n by n matrix **A**.

Definition

If $\mathbf{A}$ and $\mathbf{B}$ are matrices with the property that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$ then $\mathbf{A}$ and $\mathbf{B}$ are *inverses*.

Example:

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 3 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$\mathbf{AB}$ is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Similarly, $\mathbf{BA} = \mathbf{I}$.
If $\mathbf{B}$ is the inverse of $\mathbf{A}$, we will indicate this with the notation:

$$\mathbf{B} = \mathbf{A}^{-1}$$

$$\mathbf{A} = \mathbf{B}^{-1}$$

Computation of Matrix Inverses
Matrix Reduction Method

$$\text{Let } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$\text{Let } \mathbf{E}_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \leftarrow (j^{th}\ row)$$

$$\text{Then } \mathbf{AE}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = j^{th} \text{ column of } \mathbf{A}$$

If $\mathbf{X}_j$ represents the j^{th} column of the inverse matrix $\mathbf{A}^{-1}$, then

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{E}_j &= \mathbf{X}_j \\ \mathbf{A}\,\mathbf{A}^{-1}\mathbf{E}_j &= \mathbf{A}\mathbf{X}_j \\ \mathbf{E}_j &= \mathbf{A}\mathbf{X}_j \end{aligned}$$

Thus, the solution to the equation $\mathbf{A}\mathbf{X} = \mathbf{E}_j$ is the j^{th} column of $\mathbf{A}^{-1}$.

$$\left(\mathbf{A} \quad \middle| \quad \mathbf{E}_j\right)$$

$$\left(\mathbf{I} \quad \middle| \quad \mathbf{X}_j\right)$$

For the 2 by 2 case, we compute the first column of the inverse of

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

by solving the equation $\mathbf{A}\mathbf{X} = \binom{1}{0}$
We do this by reducing the augmented matrix

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 0 \end{array}\right)$$

to the reduced matrix

$$\left(\begin{array}{cc|c} 1 & 0 & b_{11} \\ 0 & 1 & b_{21} \end{array}\right)$$

The last column will be the first column of $\mathbf{A}^{-1}$

Similarly, we compute the second column of $\mathbf{A}^{-1}$ by solving $\mathbf{AX} = \binom{0}{1}$. We do this by reducing the augmented matrix:

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \end{array}\right)$$

to the reduced form:

$$\left(\begin{array}{cc|c} 1 & 0 & b_{12} \\ 0 & 1 & b_{22} \end{array}\right)$$

The last column will be the second column of $\mathbf{A}^{-1}$.

Combine

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 0 \end{array}\right)$$

together with

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \end{array}\right)$$

to form the augmented matrix

$$\left(\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right)$$

Now, reduce this until you obtain the reduced matrix:

$$\left(\begin{array}{cc|cc} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{array}\right)$$

Example:

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad \text{Find } \mathbf{A}^{-1}$$

Solution: Form the augmented matrix

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ will be the inverse of $\mathbf{A}$.

Example:

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix} \qquad \text{Find } \mathbf{A}^{-1}$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 3 & 10 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 10 & -3 \\ 0 & 1 & -3 & 1 \end{array}\right)$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 10 & -3 \\ -3 & 1 \end{pmatrix}$$

$$\begin{array}{ccccc} x & + & 3y & = & 2 \\ 3x & + & 10y & = & 3 \end{array}$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\mathbf{AX} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{X} \\ &= \mathbf{A}^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ -3 \end{pmatrix} \end{aligned}$$

Thus, $x = 11$ and $y = -3$ are the solutions.