# Matrix Differential Equations
## Jacobs

One of the very interesting lessons in this course is how certain algebraic techniques can be used to solve differential equations. The purpose of these notes is to describe how the solution $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of the matrix equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ will apply to the solution of the differential equation $a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = 0$. As we will see later, the differential equation can be rewritten in a matrix form and then the eigenvectors and eigenvalues of the matrix then lead to a solution.

### Review of Eigenvectors and Eigenvalues

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\vec{\mathbf{u}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. A number $\lambda$ is said to be an *eigenvalue* of $\mathbf{A}$ if there is a nonzero vector $\vec{\mathbf{u}}$ so that $\mathbf{A}\vec{\mathbf{u}} = \lambda\vec{\mathbf{u}}$. The vector $\vec{\mathbf{u}}$ is said to be an *eigenvector* of $\mathbf{A}$. Please note that the equation $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{0}}$, where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\vec{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, is completely equivalent to the equation $\mathbf{A}\vec{\mathbf{u}} = \lambda\vec{\mathbf{u}}$. This is important because if we seek a nonzero solution $\vec{\mathbf{u}}$ of $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{0}}$ then the matrix $\mathbf{A} - \lambda\mathbf{I}$ had better not have an inverse. After all, if $\mathbf{A} - \lambda\mathbf{I}$ had an inverse, then the equation $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{0}}$ could be solved by multiplying both sides by this inverse and we would obtain $\vec{\mathbf{u}} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\vec{\mathbf{0}} = \vec{\mathbf{0}}$. This could not be if $\vec{\mathbf{u}}$ is supposed to be a nonzero solution of the equation.

If $\mathbf{A} - \lambda\mathbf{I}$ has no inverse then the determinant of $\mathbf{A} - \lambda\mathbf{I}$ must be 0, and this is how we find the eigenvalues.

**Example:** Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

The determinant of $\mathbf{A} - \lambda\mathbf{I}$ is $\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6$. This is zero only when $\lambda = 2$ or $\lambda = 3$, so these are the eigenvalues.

To find the eigenvectors corresponding to $\lambda = 2$, we solve $(\mathbf{A} - 2\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{0}}$

$$(\mathbf{A} - 2\mathbf{I})\vec{\mathbf{u}} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If we multiply out and compare coordinates, we get $-u_1 + 2u_2 = 0$ so $u_1 = 2u_2$

$$\vec{\mathbf{u}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix} = u_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus, any nonzero scalar multiple of $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ will be an eigenvector corresponding to eigenvalue $\lambda = 2$.

Next, we find the eigenvectors corresponding to $\lambda = 3$ by solving the matrix equation $(\mathbf{A} - 3\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{0}}$

$$(\mathbf{A} - 3\mathbf{I})\vec{\mathbf{u}} = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This implies that $\vec{\mathbf{u}}$ is a solution only if $-u_1 + u_2 = 0$

$$\vec{\mathbf{u}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_2 \end{pmatrix} = u_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore, any nonzero scalar multiple of $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ will be an eigenvector corresponding to eigenvalue $\lambda = 3$.

### Diagonal Matrices

A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is said to be *diagonal* if $b = c = 0$. So, for example, the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a diagonal matrix. Just about any matrix calculation in easy to do with diagonal matrices. For example, look at the following matrix multiplication, where the matrices are not diagonal:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Compare this to the simplicity of multiplying two diagonal matrices:

$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & 0 \\ 0 & a_{22}b_{22} \end{pmatrix}$$

Finding the inverse of a matrix is also simpler if the matrix is diagonal. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has an inverse, then it is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Notice how much simpler this formula is if the matrix is diagonal:

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$$

### Diagonalization of Matrices

In many cases, we can take matrices that are not diagonal and put them in terms of a diagonal matrix through a simple matrix multiplication formula. As a simple demonstration, take the matrix $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 \\ -1 & 4 \end{smallmatrix}\right)$ whose eigenvectors $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ have already been calculated. We begin by constructing a matrix $\mathbf{P} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ from these eigenvectors. Now, look what happens when we calculate the matrix product $\mathbf{P}^{-1}\mathbf{AP}$.

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{AP} &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \end{aligned}$$

We have just obtained a diagonal matrix. This formula applies to a more general matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Suppose the eigenvalues of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are $\lambda_1$ and $\lambda_2$ with eigenvectors $\vec{\mathbf{u}}_1 = \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}$ and $\vec{\mathbf{u}}_2 = \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix}$ respectively. This means that:

$$\mathbf{A}\vec{\mathbf{u}}_1 = \lambda_1\vec{\mathbf{u}}_1 \qquad\qquad \mathbf{A}\vec{\mathbf{u}}_2 = \lambda_2\vec{\mathbf{u}}_2$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix} = \begin{pmatrix} \lambda_1 p_{11} \\ \lambda_1 p_{21} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix} = \begin{pmatrix} \lambda_2 p_{12} \\ \lambda_2 p_{22} \end{pmatrix}$$

Now, let's combine these results. Let $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$

$$\mathbf{AP} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} \\ \lambda_1 p_{21} & \lambda_2 p_{22} \end{pmatrix}$$

Compare this to the product $\mathbf{P\Lambda}$ where $\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\mathbf{P\Lambda} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} \\ \lambda_1 p_{21} & \lambda_2 p_{22} \end{pmatrix}$$

We see that $\mathbf{AP} = \mathbf{P\Lambda}$. This means that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{\Lambda}$ and therefore $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. Of course, this assumes that the matrix $\mathbf{P}$ has an inverse. It is easy to prove that as long as $\lambda_1 \neq \lambda_2$, then the columns of $\mathbf{P}$ are linearly independent and $\mathbf{P}$ will have an inverse.

**Example**: Let $\mathbf{A}= \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix}$. Find a matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.

We begin by solving $0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 6\lambda + 8$ to get the eigenvalues.

$$\lambda_1 = 2 \qquad\qquad \lambda_2 = 4$$

The corresponding eigenvectors are:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

and therefore, the required matrix $\mathbf{P}$ is

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$$

Let's check this answer:

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\end{aligned}$$

The result is a diagonal matrix. Notice that the entries along the diagonal are exactly the eigenvalues of $\mathbf{A}$.

**Application to Systems of Linear Differential Equations**

Problems involving interconnecting spring systems, tank systems and electric circuits involve several unknown functions solving a system of differential equations.

Let's consider, for example the problem of finding two functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$ that solve the following system of differential equations:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) \end{aligned}$$

Let's put this in matrix notation. Let $\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ so $\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$ and let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The system of differential equations can now be written as $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$. The trick to solving this equation is to perform a change of variable that transforms this differential equation into one involving only a diagonal matrix.

Using the eigenvector procedure, we can find a matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. We can define a vector-valued function $\vec{\mathbf{v}} = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$ by the formula $\vec{\mathbf{v}} = \mathbf{P}^{-1}\vec{\mathbf{x}}$. Make the substitution $\vec{\mathbf{x}} = \mathbf{P}\vec{\mathbf{v}}$ into the differential equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$.

$$\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$$

$$\frac{d}{dt}\left(\mathbf{P}\vec{\mathbf{v}}\right) = \mathbf{A}\vec{\mathbf{x}} = \mathbf{A}\mathbf{P}\vec{\mathbf{v}}$$

$$\mathbf{P}\frac{d\vec{\mathbf{v}}}{dt} = \mathbf{A}\mathbf{P}\vec{\mathbf{v}}$$

$$\frac{d\vec{\mathbf{v}}}{dt} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\vec{\mathbf{v}}$$

Since $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix, the matrix differential equation is now:

$$\begin{pmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 \\ \lambda_2 v_2 \end{pmatrix}$$

If we now compare coordinates, we get two simple differential equations:

$$\frac{dv_1}{dt} = \lambda_1 v_1 \qquad\qquad \frac{dv_2}{dt} = \lambda_2 v_2$$

These equations can be solved easily using separation of variables.

$$v_1(t) = c_1 e^{\lambda_1 t} \qquad\qquad v_2(t) = c_2 e^{\lambda_2 t}$$

where $c_1$ and $c_2$ are constants. Now that we have the coordinates of $\vec{\mathbf{v}}$, we can obtain the coordinates of $\vec{\mathbf{x}}$ from the equation $\vec{\mathbf{x}} = \mathbf{P}\vec{\mathbf{v}}$.

**Example:**

Solve the following system of differential equations:

$$\begin{aligned} x_1'(t) &= \phantom{-}x_1(t) + 2x_2(t) \\ x_2'(t) &= -x_1(t) + 4x_2(t) \end{aligned}$$

In matrix form this equation is $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 \\ -1 & 4 \end{smallmatrix}\right)$. For this matrix, we have already found $\mathbf{P} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ so if we make the substitution

$\vec{\mathbf{x}} = \mathbf{P}\vec{\mathbf{v}}$ into the equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$, we will get $\frac{d\vec{\mathbf{v}}}{dt} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)\vec{\mathbf{v}}$ whose solution is $\vec{\mathbf{v}} = \left(\begin{smallmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{smallmatrix}\right)$. The solution we seek is:

$$\vec{\mathbf{x}} = \mathbf{P}\vec{\mathbf{v}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} 2c_1 e^{2t} + c_2 e^{3t} \\ c_1 e^{2t} + c_2 e^{3t} \end{pmatrix}$$

Comparing coordinates, we get:

$$x_1(t) = 2c_1 e^{2t} + c_2 e^{3t} \text{ and } x_2(t) = c_1 e^{2t} + c_2 e^{3t}$$

There is also a useful vector format for the answer if we split up the vector solution as follows:

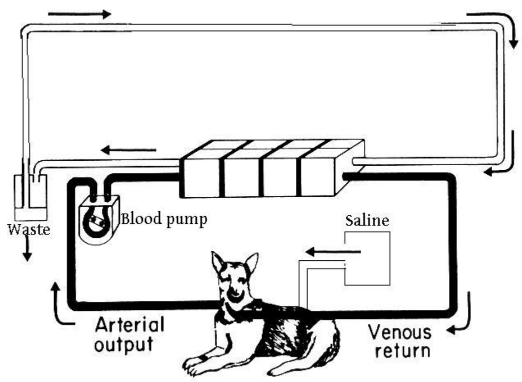
$$\vec{\mathbf{x}} = \begin{pmatrix} 2c_1 e^{2t} + c_2 e^{3t} \\ c_1 e^{2t} + c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} 2c_1 e^{2t} \\ c_1 e^{2t} \end{pmatrix} + \begin{pmatrix} c_2 e^{3t} \\ c_2 e^{3t} \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

Notice that the general solution is a linear combination of terms of the form $\vec{\mathbf{u}}e^{rt}$ where $r$ is an eigenvalue and $\vec{\mathbf{u}}$ is the corresponding eigenvector. This observation can speed up the solution process.

**Example:**

An animal is receiving medication from an external drug recycling system at an animal hospital. There are 3 liters of blood in this animal. Fluid is being delivered into the animal intravenously. The fluid, containing both blood and the drug, is entering the animal at the rate of $\frac{1}{8}$ liters/hour. A saline solution, containing no drug at all, is entering the animal at the rate of $\frac{1}{4}$ liters/hour. Blood is being drawn from the animal and sent back to the external system at the rate of $\frac{3}{8}$ liters/hour. Fluid is also being drawn out of the external system and into a waste receptacle at the rate of $\frac{1}{4}$ liters/hour. The external system has 2 liters of fluid altogether and this volume stays constant.

Let $x(t)$ be the number of milligrams of drug in the animal after $t$ hours. Let $y(t)$ be the number of milligrams of drug in the external system after $t$ hours.




Assume an initial condition of $x(0) = 0$ and $y(0) = 210$ milligrams of drug. Find the formulas for $x(t)$ and $y(t)$ by solving the appropriate system of differential equations.

$$\frac{dx}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = \frac{1}{8}\,\frac{\text{liter}}{\text{hour}} \cdot \frac{y}{2}\,\frac{\text{mg}}{\text{liter}} - \frac{3}{8}\,\frac{\text{liter}}{\text{hour}} \cdot \frac{x}{3}\,\frac{\text{mg}}{\text{liter}}$$

$$\frac{dy}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = \frac{3}{8}\,\frac{\text{liter}}{\text{hour}} \cdot \frac{x}{3}\,\frac{\text{mg}}{\text{liter}} - \frac{3}{8}\,\frac{\text{liter}}{\text{hour}} \cdot \frac{y}{2}\,\frac{\text{mg}}{\text{liter}}$$

More concisely:

$$\begin{aligned} x' &= -\frac{1}{8}x + \frac{1}{16}y \\ y' &= \ \ \frac{1}{8}x - \frac{3}{16}y \end{aligned}$$

Or, in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1/8 & 1/16 \\ 1/8 & -3/16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If $\vec{\mathbf{x}} = \left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right)$ and $\mathbf{A} = \left(\begin{smallmatrix} -1/8 & 1/16 \\ 1/8 & -3/16 \end{smallmatrix}\right)$. We can now solve $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$. We begin with the eigenvalues by solving:

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -1/8 - \lambda & 1/16 \\ 1/8 & -3/16 - \lambda \end{vmatrix} = \lambda^2 + \frac{5}{16}\lambda + \frac{1}{64}$$

After factoring:

$$0 = \left(\lambda + \frac{1}{4}\right)\left(\lambda + \frac{1}{16}\right)$$

So, the eigenvalues are:

$$\lambda = -\frac{1}{4} \qquad\qquad \lambda = -\frac{1}{16}$$

The corresponding eigenvectors are:

$$\vec{\mathbf{u}}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad\qquad \vec{\mathbf{u}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If we proceed just as we did in the last example, we end up with the general solution:

$$\vec{\mathbf{x}} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/16}$$

We have an initial condition $\vec{\mathbf{x}}(0) = \left(\begin{smallmatrix} 0 \\ 210 \end{smallmatrix}\right)$ which implies that $c_1 = -70$ and $c_2 = 70$ so

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -70 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4} + 70 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/16}$$

Equating coordinates:

$$x(t) = 70\left(-e^{-t/4} + e^{-t/16}\right) \qquad\qquad y(t) = 70\left(2e^{-t/4} + e^{-t/16}\right)$$

**Example:** Solve the equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ and $\vec{\mathbf{x}}$ are defined as:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \qquad \vec{\mathbf{x}} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

We begin with the eigenvalues of $\mathbf{A}$ which we get by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

Fortunately, this factors nicely

$$-(\lambda - 1)(\lambda + 1)(\lambda - 2) = 0$$

Therefore the eigenvalues are:

$$\lambda = 1 \qquad \lambda = -1 \qquad \lambda = 2$$

and, if you calculate the corresponding eigenvectors, you will get:

$$\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{\mathbf{u}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

If the format from the previous example also applies to this three dimensional case, we would expect the general solution to be a linear combination of all solutions of the form $\vec{\mathbf{u}}e^{rt}$ which would give us:

$$\vec{\mathbf{x}} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{1t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-1t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t}$$

Will this really be true? Again, the diagonalization method comes to the rescue. We form the matrix $\mathbf{P}$ with the eigenvectors making up the columns.

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix}$$

Now, we can calculate $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ and verify that it is diagonal.

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 1 & 1/2 & -1/2 \\ 1/3 & -1/2 & 1/6 \\ -1/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\end{aligned}$$

If we now define $\vec{\mathbf{v}} = \mathbf{P}^{-1}\vec{\mathbf{x}}$ as before, the equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ transforms to the equation $\frac{d\vec{\mathbf{v}}}{dt} = \mathbf{\Lambda}\vec{\mathbf{v}}$ where $\mathbf{\Lambda}$ is a diagonal matrix.

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_2 \\ 2v_3 \end{pmatrix}$$

Equate coordinates:

$$\frac{dv_1}{dt} = v_1 \qquad \frac{dv_2}{dt} = -v_2 \qquad \frac{dv_3}{dt} = 2v_3$$

$$v_1 = c_1 e^t \qquad v_2 = c_2 e^{-t} \qquad v_3 = c_3 e^{2t}$$

$$\vec{\mathbf{x}} = \mathbf{P}\vec{\mathbf{v}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 e^{-t} \\ c_3 e^{2t} \end{pmatrix}$$
$$= \begin{pmatrix} c_1 e^t + c_2 e^{-t} + c_3 e^{2t} \\ c_1 e^t - c_2 e^{-t} + 2c_3 e^{2t} \\ c_1 e^t + c_2 e^{-t} + 4c_3 e^{4t} \end{pmatrix}$$
$$= c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{1t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-1t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t}$$

This verifies the solution we had assumed at the outset.

**The Repeated Root Case**

Our solution of $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ depended on being able to find a matrix an invertible matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. This cannot always be done.

Consider, for example, $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ when $\vec{\mathbf{x}} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. The first step is to find the eigenvalues by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) = 0$$

This time, we only get one distinct eigenvalue, $\lambda = 2$. This is an example of the *repeated root case*. The eigenvectors consist of all nonzero scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $\mathbf{P} = \begin{pmatrix} 1 & ? \\ 0 & ? \end{pmatrix}$. We don't have another eigenvector to put in the second column of $\mathbf{P}$. If we take some scalar multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to be the second column, then $\mathbf{P} = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}$ will not have an inverse and we still can't calculate $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Fortunately, we can still solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$ a different way, so we can see what the solution looks like.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ 2x_2 \end{pmatrix}$$

Comparing coordinates:

$$\frac{dx_1}{dt} = 2x_1 + x_2 \qquad\qquad \frac{dx_2}{dt} = 2x_2$$

The solution of $\frac{dx_2}{dt} = 2x_2$ is $x_2(t) = be^{2t}$, where $b$ is a constant. If we substitute this into $\frac{dx_1}{dt} = 2x_1 + x_2$, we get the equation $\frac{dx_1}{dt} = 2x_1 + be^{2t}$ which can be solved using an integrating factor.

The solution of $\frac{dx_1}{dt} = 2x_1 + be^{2t}$ is $x_1(t) = ae^{2t} + bte^{2t}$ where $a$ is another constant. Put these coordinates into the vector $\vec{\mathbf{x}}$

$$\vec{\mathbf{x}} = \begin{pmatrix} ae^{2t} + bte^{2t} \\ be^{2t} \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + b\begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t}$$

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$ is of the form $\vec{\mathbf{u}}e^{rt}$, where $\vec{\mathbf{u}}$ is an eigenvector, but the vector $\begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t}$ does not have this form.

**Nonhomogeneous Equations**

If $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal then the equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ is easily solved. The equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ is referred to as a *homogeneous* differential equation. We can also solve equations of the form $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$ (a *nonhomogeneous* differential equation). Let's use $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ because we have already discovered that if $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ then $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Let $\vec{\mathbf{f}} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$.

As usual, we make the substitution $\vec{\mathbf{x}} = \mathbf{P}\vec{\mathbf{v}}$ into the equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$

$$\frac{d}{dt}(\mathbf{P}\vec{\mathbf{v}}) = \mathbf{AP}\vec{\mathbf{v}} + \vec{\mathbf{f}}$$

$$\mathbf{P}\frac{d\vec{\mathbf{v}}}{dt} = \mathbf{AP}\vec{\mathbf{v}} + \vec{\mathbf{f}}$$

$$\frac{d\vec{\mathbf{v}}}{dt} = \mathbf{P}^{-1}\mathbf{AP}\vec{\mathbf{v}} + \mathbf{P}^{-1}\vec{\mathbf{f}}$$

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2v_1 + e^{2t} \\ 3v_2 - e^{2t} \end{pmatrix}$$

Comparing coordinates, we get:

$$\frac{dv_1}{dt} = 2v_1 + e^{2t} \qquad\qquad \frac{dv_2}{dt} = 3v_2 - e^{2t}$$

Solve each of these using integrating factors.

$$v_1 = c_1e^{2t} + te^{2t} \qquad\qquad v_2 = c_2e^{3t} + e^{2t}$$

Now we can substitute into $\mathbf{P}\vec{\mathbf{v}}$ and get the solution $\vec{\mathbf{x}}$

$$\vec{\mathbf{x}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1e^{2t} + te^{2t} \\ c_2e^{3t} + e^{2t} \end{pmatrix}$$

If we multiply out and regroup we can write the solution in the form:

$$\vec{\mathbf{x}} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 2t+1 \\ t+1 \end{pmatrix} e^{2t}$$

**Solution of Second Order Linear Differential Equations**

The matrix differential equations we have solved can be applied to equations of the form $a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = 0$. Since we can always divide both sides by $a_2$, we might as well assume that $a_2 = 1$, so the equation becomes $y'' + a_1y' + a_0y = 0$. This equation can always be converted to a matrix form. Let $x_1 = y$ and $x_2 = y'$. If $\vec{\mathbf{x}} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ then the derivative of $\vec{\mathbf{x}}$ is:

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -a_0y - a_1y' \end{pmatrix} = \begin{pmatrix} x_2 \\ -a_0x_1 - a_1x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We have just converted the scalar differential equation $y'' + a_1y' + a_0y = 0$ to a matrix equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$. The first coordinate of $\vec{\mathbf{x}}$ will be the solution of $y'' + a_1y' + a_0y = 0$.

**Example:**

Find the general solution $y = y(t)$ of $y'' - 6y' + 8y = 0$

If we let $\vec{\mathbf{x}} = \left(\begin{smallmatrix} y \\ y' \end{smallmatrix}\right)$ then $\frac{d\vec{\mathbf{x}}}{dt} = \left(\begin{smallmatrix} 0 & 1 \\ -8 & 6 \end{smallmatrix}\right)\vec{\mathbf{x}}$. We have already discovered that the eigenvalues of this matrix are $\lambda = 2$ and $\lambda = 4$ and the corresponding eigenvectors are $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right)$. Therefore, the solution of $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ is:

$$\vec{\mathbf{x}} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t} = \begin{pmatrix} c_1e^{2t} + c_2e^{4t} \\ 2c_1e^{2t} + 4c_2e^{4t} \end{pmatrix}$$

The solution $y = y(t)$ is the first coordinate of this vector so:

$$y = c_1 e^{2t} + c_2 e^{4t}$$

Notice that each term is a constant times $e^{rt}$ where $r$ is an eigenvalue.

**Example:**

Find the general solution of $y''' - 2y'' - y' + 2y = 0$.

This is a third order differential equation, but we can extend our matrix techniques so we can solve this too. Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$ be the coordinates of a vector $\vec{\mathbf{x}}$. Take the derivative of $\vec{\mathbf{x}}$.

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ y''' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ -2y + y' + 2y'' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ -2x_1 + x_2 + 2x_3 \end{pmatrix}$$

We can write this as a matrix multiplication.

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We have solved $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ for this particular matrix already. The general solution is:

$$\vec{\mathbf{x}} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{1t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-1t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t}$$

Now, we can extract the first coordinate of this solution to get $y(t)$

$$y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t}$$