Representation of Systems of Equations Using Matrices

Two equations, two unknowns

$$\begin{aligned} 2x + 1y &= 3 \\ 4x + 7y &= 2 \end{aligned}$$

Coefficient Matrix: $\begin{pmatrix} 2 & 1 \\ 4 & 7 \end{pmatrix}$

Augmented Matrix: $\left(\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 7 & 2 \end{array}\right)$

The notation $\left(\begin{array}{cc|c} a & b & u \\ c & d & v \end{array}\right)$

represents the system of equations

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}$$

Operations With Systems of Equations

1. Multiply both sides of an equation by a non-zero constant.

$$\begin{array}{rcrcr} -2x & + & 3y & = & 1 \\ x & + & y & = & 2 \end{array} \qquad \left(\begin{array}{rr|r} -2 & 3 & 1 \\ 1 & 1 & 2 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{array}{rcrcr} -2x & + & 3y & = & 1 \\ 2x & + & 2y & = & 4 \end{array} \qquad \left(\begin{array}{rr|r} -2 & 3 & 1 \\ 2 & 2 & 4 \end{array}\right)$$

2. Interchange two equations

$$\begin{array}{rcrcl} -2x & + & 3y & = & 1 \\ 2x & + & 2y & = & 4 \end{array} \qquad \left(\begin{array}{rr|r} -2 & 3 & 1 \\ 2 & 2 & 4 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{array}{rcrcl} 2x & + & 2y & = & 4 \\ -2x & + & 3y & = & 1 \end{array} \qquad \left(\begin{array}{rr|r} 2 & 2 & 4 \\ -2 & 3 & 1 \end{array}\right)$$

3. Add a multiple of one equation to another

$$\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ -2x & + & 3y & = & 1 \end{array} \qquad \left(\begin{array}{rr|r} 2 & 2 & 4 \\ -2 & 3 & 1 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ 0x & + & 5y & = & 5 \end{array} \qquad \left(\begin{array}{rr|r} 2 & 2 & 4 \\ 0 & 5 & 5 \end{array}\right)$$

Elementary Row Operations

1. Multiply a row through by a non-zero constant
2. Interchange two rows
3. Add a multiple of one row to another.

Solutions of a System of Equations Using Elementary Row Operations

Matrix	System of Equations
$\left(\begin{array}{rr\|r} -2 & 3 & 1 \\ 1 & 1 & 2 \end{array}\right)$	$\begin{array}{rcrcr} -2x & + & 3y & = & 1 \\ x & + & y & = & 2 \end{array}$

$$\Big\downarrow \text{ Replace } R_2 \text{ with } 2R_2$$

$\left(\begin{array}{rr\|r} -2 & 3 & 1 \\ 2 & 2 & 4 \end{array}\right)$	$\begin{array}{rcrcr} -2x & + & 3y & = & 1 \\ 2x & + & 2y & = & 4 \end{array}$

$$\Big\downarrow \text{ Interchange } R_1 \text{ and } R_2$$

$\left(\begin{array}{rr\|r} 2 & 2 & 4 \\ -2 & 3 & 1 \end{array}\right)$	$\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ -2x & + & 3y & = & 1 \end{array}$

$$\Big\downarrow \text{ Replace } R_2 \text{ with } R_1 + R_2$$

$\left(\begin{array}{rr\|r} 2 & 2 & 4 \\ 0 & 5 & 5 \end{array}\right)$	$\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ 0x & + & 5y & = & 5 \end{array}$

$$\Big\downarrow \text{ Replace } R_1 \text{ with } \tfrac{1}{2}R_1 \text{ and}$$

$$\Big\downarrow \text{ Replace } R_2 \text{ with } \tfrac{1}{5}R_2$$

$\left(\begin{array}{rr\|r} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right)$	$\begin{array}{rcrcr} x & + & y & = & 2 \\ 0x & + & 1y & = & 1 \end{array}$

$$\Big\downarrow \text{ Replace } R_1 \text{ with } R_1 - R_2$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{rcrcr} x & + & y & = & 2 \\ 0x & + & 1y & = & 1 \end{array}$$

$$\Big\downarrow \text{ Replace } R_1 \text{ with } R_1 - R_2$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{rcrcr} 1x & + & 0y & = & 1 \\ 0x & + & 1y & = & 1 \end{array}$$

The final matrix is in *Reduced Row Echelon Form*
The process demonstrated above is *Matrix Reduction*
The use of matrix reduction to reduce a matrix to reduced row echelon form is *Gauss-Jordan Elimination*

Echelon Form

1. If a row does not consist entirely of zeros then the first nonzero number in the row is a 1

2. If there are any rows that consist entirely of zeros then they are grouped together at the bottom of the matrix

3. In any 2 successive rows that do not consist entirely of zeros, the leading 1 in the lower row is further to the right than the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeros everywhere else.

A matrix that has properties 1 - 3 is in *Row Echelon Form*

A matrix that has properties 1 - 4 is in *Reduced Row Echelon Form*

$$
\begin{aligned}
x + y + z &= 1 \\
x + 2y - z &= 0 \\
x - y - z &= 0
\end{aligned}
$$

$$
\left(\begin{array}{rrr|r}
1 & 1 & 1 & 1 \\
1 & 2 & -1 & 0 \\
1 & -1 & -1 & 0
\end{array}\right)
$$

$$
\Big\downarrow \begin{array}{l} \textit{Replace } R_2 \textit{ with } -R_1 + R_2 \\ \textit{Replace } R_3 \textit{ with } -R_1 + R_3 \end{array}
$$

$$
\left(\begin{array}{rrr|r}
1 & 1 & 1 & 1 \\
0 & 1 & -2 & -1 \\
0 & -2 & -2 & -1
\end{array}\right)
$$

$$
\Big\downarrow \textit{Replace } R_3 \textit{ with } 2R_2 + R_3
$$

$$
\left(\begin{array}{rrr|r}
1 & 1 & 1 & 1 \\
0 & 1 & -2 & -1 \\
0 & 0 & -6 & -3
\end{array}\right)
$$

$$
\Big\downarrow \textit{Replace } R_3 \textit{ with } -\tfrac{1}{6}R_3
$$

$$
\left(\begin{array}{rrr|r}
1 & 1 & 1 & 1 \\
0 & 1 & -2 & -1 \\
0 & 0 & 1 & \frac{1}{2}
\end{array}\right)
$$

$$
\Big\downarrow \textit{Replace } R_2 \textit{ with } R_2 + 2R_3
$$

$$
\left(\begin{array}{ccc|c}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \frac{1}{2}
\end{array}\right)
\Bigg\downarrow \textit{Replace } R_1 \textit{ with } R_1 - R_2 - R_3
\left(\begin{array}{ccc|c}
1 & 0 & 0 & \frac{1}{2} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \frac{1}{2}
\end{array}\right)
$$

Reduced Row Echelon Form

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 0 \\
x_1 + 3x_2 + x_3 &= 0 \\
x_1 + x_2 - 3x_3 &= 0
\end{aligned}
$$

$$
\left(\begin{array}{ccc|c}
1 & 2 & 1 & 0 \\
1 & 3 & 1 & 0 \\
1 & 1 & -3 & 0
\end{array}\right)
$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right)$$

$$\textit{Replace } R_2 \textit{ with } -R_1 + R_2 \Big\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right)$$

$$\textit{Replace } R_3 \textit{ with } -R_1 + R_3 \Big\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right)$$

$$Replace\ R_3\ with\ R_2 + R_3 \Big\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$Replace\ R_1\ with\ -2R_2 + R_1 \Big\downarrow$$

$$Replace\ R_3\ with\ -\tfrac{1}{4}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Replace R_1 *with* $-R_3$ *added to* R_1 $\big\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This matrix represents the following system of equations:

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 1x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 1x_3 &= 0 \end{aligned}$$

Conclusion: $x_1 = x_2 = x_3 = 0$