Series Solutions of Differential Equations
Jacobs

Introduction.

The methods you have learned to solve the linear differential equation $P(D)y = 0$ work very well as long as the operator $P(D)$ have constant coefficients. For example:

$$y'' - 2y' = 0$$

The standard technique we have used is to substitute e^{rx} (where r is a constant) into the differential equation.

$$r^2 e^{rx} - 2re^{rx} = 0$$

$$r^2 - 2r = 0$$

$$\text{Solutions:} \quad r = 0 \text{ and } \quad r = 2$$

Therefore, the general solution is:

$$y = ae^{0x} + be^{2x} = a + be^{2x}$$

Things do not work out as well if there are variables in the coefficients. For example:

$$y'' - 2xy' = 0$$

Let's see what happens if we try substituting e^{rx}? Once again, r is supposed to be a constant.

$$r^2 e^{rx} - 2xre^{rx} = 0$$

$$r^2 - 2xr = 0$$

$$r(r - 2x) = 0$$

We do get $r = 0$ for one solution, but $r = 2x$ would be a contradiction because r is supposed to be a constant. The equation $y'' - 2xy' = 0$ is a second order equation and theory tells us that there are supposed to be two linearly independent solutions. Substituting e^{rx} didn't give us both of them. In general, if there are variables in the coefficients, we may not get *any* solutions of the form e^{rx}. What do we do?

We can actually get a little further with the equation $y'' - 2xy' = 0$ if we use a derivative formula from first year calculus. Take the formula for the derivative of the log of a function:

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$$

If we let $f(x) = y'$ then we get:

$$\frac{d}{dx}(\ln y') = \frac{y''}{y'}$$

Now, back to the equation $y'' - 2xy' = 0$. This can be rewritten as:

$$\frac{y''}{y'} = 2x$$

$$\frac{d}{dx}\ln(y') = 2x$$

$$\ln(y') = x^2 + C$$

$$y' = e^{x^2+C} = e^C e^{x^2} = ke^{x^2} \quad \text{where} \quad k = e^C$$

We haven't found the solution yet, but we do have its derivative

$$y' = ke^{x^2}$$

All we have to do is integrate.

$$y = k\int e^{x^2}\,dx$$

This is not an easy integral. There is a way to express the answer as an infinite series.

Review of Power Series

Back in Calculus II, you learned that under certain differentiability conditions, a function $f(x)$ could be expressed as a power series:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

Or, in summation form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Furthermore, you calculated power series for a variety of functions back in MA 242:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

In particular, look what we get if we take the x in the series for e^x and replace it with x^2.

$$\text{If} \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{then} \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x^2\right)^n$$

So the solution to $y'' - 2xy' = 0$ is:

$$y = k \int e^{x^2}\, dx = k \int \left(1 + \frac{x^2}{1} + \frac{x^4}{2} + \cdots\right) dx$$

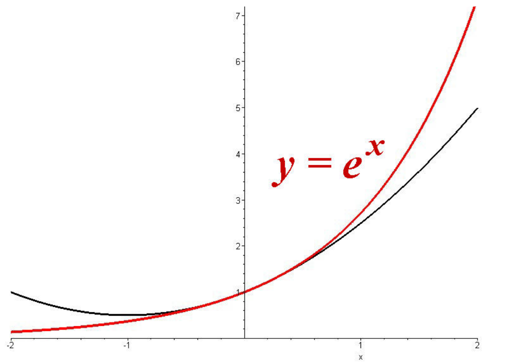
This isn't bad, because even though the sum goes on for ever, we are still integrating a big polynomial. The integral of a sum is the sum of the individual integrals:

$$\begin{aligned}
y &= k \int e^{x^2}\, dx \\
&= k \int \left(1 + \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots\right) dx \\
&= k \left(x + \frac{x^3}{3} + \frac{x^5}{(2)(5)} + \frac{x^7}{(3!)(7)} + \cdots\right)
\end{aligned}$$

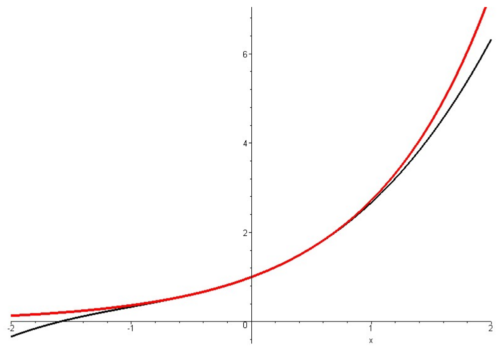
We have just expressed our solution in the form of a power series. To appreciate what's good about that, take the function $g(x) = e^x$ as an example. Suppose you didn't know that $g(x)$ was equal to e^x, but you did have it's power series:

$$1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

Let's graph e^x and $1 + x + \frac{1}{2}x^2$ on the same axis.

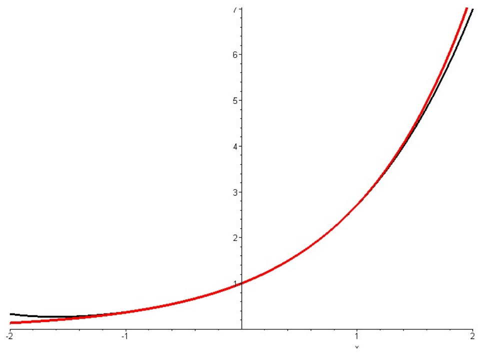


We can see from the graph that $1+x+\frac{1}{2}x^2$ is a pretty good approximation of e^x, at least when x is close to 0. Now, compare the graph of e^x to the graph of $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$



Now the approximation is even better.

Let's add another term.
Compare the graphs of e^x to $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$



The more terms we add, the closer the power series gets to e^x. If nothing else, power series gives us a great approximation tool. If we are trying to solve a differential equation but all we knew about the solution was it's power series $\sum a_n x^n$, we could still obtain polynomial approximations for the solution.

How do we find the power series $\sum a_n x^n$ for a function $y = y(x)$ if we know the differential equation that y satisfies?

Series Solution of a First Order Equation

The simplest way to find an expression of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

that solves a differential equation is to plug the sum into the equation and try to find the coefficients that make it work. Think of this as similar to the method of undetermined coefficients, but there are an infinite number of coefficients to find. For example, let's obtain a series solution of the differential equation:

$$\frac{dy}{dx} - 7y = 0 \qquad \text{where } y(0) = 1$$

First of all, if $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ then $y(0) = a_0$. Since the initial condition says that $y(0) = 1$, this is what a_0 is. We might as well plug that in:

$$y = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

One coefficient down. Only an infinite number of them to go. The rest of the coefficients can be obtained from the differential equation itself. To substitute into the differential equation, we need $\frac{dy}{dx}$.
If $y = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ then $\frac{dy}{dx}$ is:

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Now, back to the differential equation:

$$\frac{dy}{dx} - 7y = 0$$

$$(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots) - 7(1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdot) = 0$$

Now, combine and collect like terms:

$$a_1 - 7 + (2a_2 - 7a_1)x + (3a_3 - 7a_2)x^2 + (4a_4 - 7a_3)x^3 + (5a_5 - 7a_4)x^4 + \cdots = 0$$

The only way this polynomial is 0 for all x is for each coefficient to be 0.

$$a_1 - 7 = 0 \qquad 2a_2 - 7a_1 = 0 \qquad 3a_3 - 7a_2 = 0 \qquad 4a_4 - 7a_3 = 0 \qquad \text{etc}$$

The equation $a_1 - 7 = 0$ implies that $a_1 = 7$
The equation $2a_2 - 7a_1 = 0$ implies that $a_2 = \frac{7a_1}{2} = \frac{7^2}{2}$
The equation $3a_3 - 7a_2 = 0$ implies that $a_3 = \frac{7a_2}{3} = \frac{7^3}{(3)(2)}$
Similarly, $a_4 = \frac{7^4}{(4)(3)(2)} = \frac{7^4}{4!}$. The general formula seems to be:

$$a_n = \frac{7^n}{n!}$$

Therefore, the solution to our differential equation is:

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{7^n}{n!} x^n$$

This happens to be the infinite series for e^{7x}, but even if we didn't recogize that, the series could still be used to approximate the solution.

We can extend this method to higher order equations, but the calculation becomes extremely tedious unless we use a trick involving the $\sum$ sign called *index shifting.* Consider the sum for e^{x^2} that came up earlier:

$$1 + \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots$$

In $\sum$ notation, this would be expressed as:

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

However, any of the following would mean exactly the same thing:

$$\sum_{n=-1}^{\infty} \frac{x^{2(n+1)}}{(n+1)!} \qquad \sum_{n=-2}^{\infty} \frac{x^{2(n+2)}}{(n+2)!} \qquad \sum_{n=-3}^{\infty} \frac{x^{2(n+3)}}{(n+3)!}$$

For more general power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Any of the following would mean the same thing:

$$\sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=-1}^{\infty} a_{n+1} x^{n+1} \qquad \sum_{n=-2}^{\infty} a_{n+2} x^{n+2}$$

There is an elegant trick that will enable us to not worry about the lower limit of summation.

$$\text{Define } a_n = 0 \text{ when } n < 0$$

Thus, a_{-1}, a_{-2}, a_{-3} all stand for 0. With this convention,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} a_n x^n &= \cdots + a_{-3}x^{-3} + a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots \\
&= \cdots + 0x^{-3} + 0x^{-2} + 0x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots \\
&= a_0 + a_1 x + a_2 x^2 + \cdots
\end{aligned}$$

Let's use the abbreviation $\sum a_n x^n$ to mean that n ranges from $-\infty$ to ∞. In this notation, all of the following sums stand for the same thing:

$$\sum a_n x^n \qquad \sum a_{n+1} x^{n+1}$$

They all stand for $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. The same notation can be used for the derivative:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Any of the following sums would equal y'

$$\sum n a_n x^{n-1} \qquad \sum (n+1) a_{n+1} x^n \qquad \sum (n+2) a_{n+2} x^{n+1}$$

We might as well state the result for the second derivative:

$$y'' = 2a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + (5)(4)a_5 x^3 + \cdots$$

Any of the following sums would equal y''

$$\sum n(n-1) a_n x^{n-2} \qquad \sum (n+1) n a_{n+1} x^{n-1} \qquad \sum (n+2)(n+1) a_{n+2} x^n$$

Now, let's return to the differential equation we solved earlier:

$$y' - 7y = 0$$

If we substitute $\sum a_n x^n$ in place of y, we get:

$$y' - 7 \sum a_n x^n = 0$$

We have several choices to use for y', all meaning the same thing:

$$y' = \sum n a_n x^{n-1} = \sum (n+1) a_{n+1} x^n = \sum (n+2) a_{n+2} x^{n+1}$$

If we use $\sum (n+1) a_{n+1} x^n$ for y', we can collect terms very nicely.

$$\sum (n+1) a_{n+1} x^n - \sum 7 a_n x^n = 0$$

$$\sum \left((n+1)a_{n+1} - 7a_n\right) x^n = 0$$

The only way this polynomial will equal 0 for all x is for each coefficient to equal 0.

$$(n+1)a_{n+1} - 7a_n = 0$$

This is referred to as the *recurrence relation.* This enables us to get a_{n+1} as soon as we know a_n.

We already know that $a_0 = 1$ from the initial condition $y(0) = 1$ and the relation $a_{n+1} = \frac{7a_n}{n+1}$ will get us all of the other coefficients:

$$a_1 = \frac{7a_0}{1} = 7 \qquad a_2 = \frac{7a_1}{2} = \frac{7^2}{2} \qquad a_3 = \frac{7a_2}{3} = \frac{7^3}{(3)(2)}$$

We are getting the same result $a_n = \frac{7^n}{n!}$ as we had before.

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{7^n}{n!} x^n$$

We are ready to try this on a second order differential equation.

Example:

Find a series of the form $\sum a_n x^n$ that solves the following differential equation:

$$y'' - y = 0 \qquad \text{where } y(0) = 1 \text{ and } y'(0) = 1$$

As before, a_0 is determined by the initial condition $y(0) = 1$ so $a_0 = 1$. The derivative of y is $y' = a_1 + 2a_2x + 3a_3x^2 + \cdots$ so $1 = y'(0) = a_1$. Now that we have both $a_0 = 1$ and $a_1 = 1$, we can find the remaining coefficients by substituting $y = \sum a_n x^n$ into the equation $y'' - y = 0$. For y'' we have a choice to make:

$$\sum n(n-1)a_n x^{n-2} \qquad \sum (n+1)n a_{n+1} x^{n-1} \qquad \sum (n+2)(n+1)a_{n+2} x^n$$

If we choose $y'' = \sum (n+2)(n+1)a_{n+2}x^n$ into the equation, we will be able to collect the sums together easily.

$$\sum (n+2)(n+1)a_{n+2}x^n - \sum a_n x^n = 0$$

$$\sum\left((n+2)(n+1)a_{n+2}-a_n\right)x^n=0$$

Each coefficient must equal 0, so this gives us our recurrence relation:

$$(n+2)(n+1)a_{n+2}-a_n=0 \quad \text{for all } n$$

For $n \geq 0$, we can solve for a_{n+2} in terms of a_n.

$$a_{n+2}=\frac{a_n}{(n+2)(n+1)}$$

We already know $a_0=1$ and $a_1=1$.
If $n=0$, the formula $a_{n+2}=\frac{a_n}{(n+2)(n+1}$ implies $a_2=\frac{1}{(2)(1)}$
If $n=1$, we get $a_3=\frac{a_1}{(3)(2)}=\frac{1}{(3)(2)}$
If $n=2$, we get $a_4=\frac{a_2}{(4)(3)}=\frac{1}{(4)(3)(2)(1)}$
If $n=3$, we get $a_5=\frac{a_3}{(5)(4)}=\frac{1}{(5)(4)(3)(2)(1)}$
We have a predictable pattern. The general expression for a_n is:

$$a_n=\frac{1}{n!}$$

Therefore the solution to the differential equation is:

$$y=\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

Of course, this is the series for e^x.

The equation $y''-y=0$ could have been solved more easily by substituting e^{rx}. However, as we have seen earlier, we can't do this if there are any variables in the coefficients. Let's try an example like that.

Example:
Solve the differential equation:

$$y''+xy'+2y=0 \quad \text{where } y(0)=0 \text{ and } y'(0)=1$$

As before, $y(0)$ and $y'(0)$ give us the first two coefficients. In this case, $a_0=0$ and $a_1=1$.

We now substitute $\sum a_n x^n$ in place of y and $\sum(n+2)(n+1)a_{n+2}x^n$ in place of y'' in the equation $y''+xy'+2y=0$. For the middle term, we have a choice to make:

$$y' = \sum n a_n x^{n-1} = \sum (n+1)a_{n+1}x^n = \sum (n+2)a_{n+2}x^{n+1}$$

The best choice would be $y' = \sum n a_n x^{n-1}$ because this way, the middle term becomes $xy' = x\sum n a_n x^{n-1} = \sum n a_n x^n$ and the sums can then be collected together.

$$y''+xy'+2y=0$$

$$\sum (n+2)(n+1)a_{n+2}x^n + \sum n a_n x^n + 2\sum a_n x^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} + n a_n + 2a_n\right) x^n = 0$$

This can only happen if the coefficient of x^n is 0 for each n.

$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0$$

For $n \geq 0$ we get:

$$a_{n+2} = \frac{-a_n}{n+1}$$

For $n=0$, we get $a_2 = \frac{-a_0}{0+1} = 0$ For $n=1$ we get $a_3 = \frac{-a_1}{2} = -\frac{1}{2}$
For $n=2$, we get $a_4 = \frac{-a_2}{2+1} = 0$ For $n=3$ we get $a_5 = \frac{-a_3}{4} = \frac{1}{(4)(2)}$
All the coefficients with even subscripts are all coming out to be 0, so we can just focus on the coefficients with the odd subscripts:

$$a_7 = \frac{-a_5}{6} = -\frac{1}{(6)(4)(2)} \qquad a_9 = \frac{-a_7}{8} = \frac{1}{(8)(6)(4)(2)}$$

Each factor in the denominator is a multiple of 2. If we factor all the twos out we get:

$$a_7 = -\frac{1}{2^3 3!} \qquad a_9 = \frac{1}{2^4 4!}$$

The earlier coefficients have the same pattern.
$a_5 = \frac{1}{(4)(2)} = \frac{1}{2^2 2!}$, $a_3 = -\frac{1}{2} = -\frac{1}{2^1 1!}$ and $a_1 = 1 = \frac{1}{2^0 0!}$

Let's see what our solution looks like:

$$
\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\
&= a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \cdots \\
&= \frac{1}{2^0 0!} x - \frac{1}{2^1 1!} x^3 + \frac{1}{2^2 2!} x^5 - \frac{1}{2^3 3!} x^7 + \frac{1}{2^4 4!} x^9 + \cdots
\end{aligned}
$$

If we factor an x out, it becomes very easy to write the final answer in summation notation:

$$
y = x \left(\frac{1}{2^0 0!} - \frac{1}{2^1 1!} x^2 + \frac{1}{2^2 2!} x^4 - \frac{1}{2^3 3!} x^6 + \frac{1}{2^4 4!} x^8 + \cdots \right)
$$

$$
y = x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n}
$$

Example:

Find a solution of the form $\sum a_n x^n$ that solves:

$$
\left(1 + 4x^2\right) y'' + 16xy' + 8y = 0 \qquad \text{where } y(0) = 1 \text{ and } y'(0) = 0
$$

The initial conditions tell us that $a_0 = 1$ and $a_1 = 0$. Now, it's time to substitute $\sum a_n x^n$ into the differential equation so that we can find the remaining coefficients. We need to decide which format of the sum to use for y' and y''. The strategy is to choose the format that gives us expressions of the form $\sum (\quad) x^n$ where the coefficient multiplying x^n depends only on n and not on x. For this problem, it helps to multiply out the first term of the differential equation:

$$
y'' + 4x^2 y'' + 16xy' + 8y = 0
$$

For the very first term, replace y'' with $\sum (n+2)(n+1) a_{n+2} x^n$

For the second term, we're better off replacing y'' with $\sum n(n-1) a_n x^{n-2}$ because this way, $4x^2 y'' = \sum 4n(n-1) a_n x^n$.

For the third term, the best choice is $y' = \sum n a_n x^{n-1}$ because this way, $16xy' = \sum 16 n a_n x^n$

$$\sum (n+2)(n+1)a_{n+2}x^n + \sum 4n(n-1)a_n x^n + \sum 16na_n x^n + \sum 8a_n x^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} + (4n(n-1)+16n+8)a_n\right)x^n = 0$$

Each coefficient must be 0 so:

$$(n+2)(n+1)a_{n+2} + (4n(n-1)+16n+8)a_n = 0 \qquad \text{for all } n$$

The equation simplifies:

$$(n+2)(n+1)a_{n+2} + (4n^2+12n+8)a_n = 0$$

$$(n+2)(n+1)a_{n+2} + 4(n+2)(n+1)a_n = 0$$

$$a_{n+2} = -4a_n$$

Since we already know $a_0 = 1$ and $a_1 = 0$, we begin by substituting $n = 0$ into our recurrence relation $a_{n+2} = -4a_n$

For $n = 0$, $a_2 = -4a_0 = -4$ | For $n = 1$, $a_3 = -4a_1 = 0$
For $n = 2$, $a_4 = -4a_2 = 4^2$ | For $n = 3$, $a_5 = -4a_3 = 0$
For $n = 4$, $a_6 = -4a_4 = -4^3$ | For $n = 5$, $a_7 = -4a_5 = 0$

$a_n = 0$ whenever n is an odd integer, so our final answer will have the form:

$$\begin{aligned} y &= a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots \\ &= 1 - 4x^2 + 4^2x^4 - 4^3x^6 + \cdots \end{aligned}$$

It's always nice if you can write the answer in summation form:

$$y = \sum_{n=0}^{\infty} (-4)^n x^{2n}$$

We have our final answer, but we can actually go one step further in this problem and write the answer in a *closed form* (not a $\sum$ expression). To see this, you have to remember the geometric series:

$$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r} \qquad \text{as long as } |r| < 1$$

In summation form:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

The solution of our differential equation $y = \sum(-4)^n x^{2n} = \sum\left(-4x^2\right)^n$ is exactly of this form with $r = -4x^2$. Therefore, the solution of the differential equation is $y = \frac{1}{1-r}$ with $r = -4x^2$

$$y = \frac{1}{1+4x^2}$$

The restriction $|r| < 1$ mentioned above is to guarantee convergence. If $r = -4x^2$ then $4x^2 < 1$ and $-\frac{1}{2} < x < \frac{1}{2}$ is the interval in which our series solution converges. The issue of convergence is one that I have ignored in this lesson. When we substitute $\sum a_n x^n$ into a differential equation, the final answer might only converge for a restricted interval. The Ratio Test from MA 242 would be useful in finding this interval.