Variation Of Parameters
Prof. Jacobs' Classes - Spring 2020

Up to this point, you have seen how to use the Annihilator Method, combined with the Method of Undetermined Coefficients to solve equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Here's a quick example. Suppose we wanted to solve the following equation:

$$y'' + 4y = x^2$$

We know that we must first obtain the homogeneous solution:

$$y_h = c_1 \cos 2x + c_2 \sin 2x$$

and that the general solution will have the form:

$$y = y_h + y_p$$

where y_p is the particular solution. So, how do we obtain the particular solution? One way is to make a good guess. Since the right hand side of $y'' + 4y = x^2$ is a second degree polynomial, a reasonable guess would be that the particular solution is also a second degree polynomial, so the general solution will have the form:

$$y = c_1 \cos 2x + c_2 \sin 2x + a_0 + a_1 x + a_2 x^2$$

Another approach is to recognize that the annihilator of x^2 is D^3, so we look at the solutions of $D^3(D^2+4)y = -0$ that are not already solutions of $y''+4y = 0$. Either way, we still have to find the coefficients a_0, a_1 and a_2. That's where the Method of Undetermined Coefficients comes in, because we can solve for a_0, a_1 and a_2 so that:

$$(D^2 + 4)(a_0 + a_1 x + a_2 x^2) = x^2$$

This will lead to the final answer:

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{8} + \frac{1}{4}x^2$$

This is a simple method, but it has it's limitations. What if we cannot guess at the general form of the particular solution? What if we can't find the necessary annihilator? For example,

$$y'' + 4y = \ln x$$

$$y = c_1 \cos 2x + c_2 \sin 2x + y_p$$

$$y_p = \mathbf{???}$$

Or, what if we were trying to solve:

$$y'' + 4y = \sec x$$

Neither making guesses at the form of the particular solution nor trying to find annihilators seem to help. Fortunately, there is another method that works more generally called the Method of Variation of Parameters and that is the purpose of these notes.

First Order Equations

Let's turn to a first order linear equations for some insight

$$\frac{dy}{dx} + 2y = f(x)$$

Whatever the formula for $f(x)$ is we know what the homogeneous solution is because we could use separation of variables on $\frac{dy}{dx} + 2y = 0$

$$y_h = ce^{-2x}$$

Now, multiply both sides of $\frac{dy}{dx} + 2y = f(x)$ by the integrating factor $\mu = e^{2x}$

$$\frac{dy}{dx} + 2y = f(x)$$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}f(x)$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}f(x)$$

$$e^{2x}y = \int e^{2x} f(x)\, dx$$

$$y = \left(\int e^{2x} f(x)\, dx\right) e^{-2x}$$

Of course, we can't proceed any further without knowing what the $f(x)$ is. Nevertheless, if we let $v(x) = \int e^{2x} f(x)\, dx$, we have just obtained a solution of the form:

$$y = v(x)e^{-2x}$$

So, ce^{-2x} (where c is a constant) is the homogeneous solution and $v \cdot e^{-2x}$ (where v is a *function*) is the nonhomogeneous solution. This turns out to generalize to higher order equation.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Suppose the homogeneous solution is

$$y = c_1 y_{1h} + c_2 y_{2h} \qquad \text{where } c_1 \text{ and } c_2 \text{ are constants}$$

It is possible to find functions

$$v_1 = v_1(x) \qquad \text{and} \qquad v_2 = v_2(x)$$

so that the solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

is:

$$y = v_1 y_{1h} + v_2 y_{2h}$$

This is the method of *variation of parameters.*

Formulas for v_1 and v_2

So, how do get these functions v_1 and v_2? It is a straightforward but somewhat tedious process to substitute $v_1 y_{1h} + v_2 y_{2h}$ into the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ and solve for v_1 and v_2. This is done in your textbook and I am not going to repeat the calculation here. However, to obtain the formulas for v_1 and v_2 we use the following formulas for their *derivatives.*

$$\frac{dv_1}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} 0 & y_{2h} \\ \frac{f(x)}{a} & y'_{2h} \end{vmatrix} \qquad \frac{dv_2}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} y_{1h} & 0 \\ y'_{1h} & \frac{f(x)}{a} \end{vmatrix}$$

where $\mathcal{W}$ is a determinant given by:

$$\mathcal{W} = \begin{vmatrix} y_{1h} & y_{2h} \\ y'_{1h} & y'_{2h} \end{vmatrix}$$

$\mathcal{W}$ is called the Wronskian determinant. As long as the homogeneous solutions y_{1h} and y_{2h} are linearly independent, $\mathcal{W}$ will be nonzero.

Without loss of generality, we may assume that $a = 1$ because we can always divide across by the leading coefficient. So, for example, if we started with the equation $4y'' - 8y = \sin x$ (where $a = 4$), we could divide by 4 to obtain $y'' - 2y = \frac{1}{4}\sin x$ (and now $a = 1$). So, if $a = 1$, the formulas simplify just a little bit:

$$\frac{dv_1}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} 0 & y_{2h} \\ f(x) & y'_{2h} \end{vmatrix} \qquad \frac{dv_2}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} y_{1h} & 0 \\ y'_{1h} & f(x) \end{vmatrix}$$

Let's try this method out on the equation $y'' + 4y = x^2$. We begin with the homogeneous solutions $y_{1h} = \cos 2x$ and $y_{2h} = \sin 2x$. The variation of parameters method will produce two functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ so that $y = v_1 \cos 2x + v_2 \sin 2x$ will be the nonhomogeneous solution. To use the formulas, we will need the Wronskian determinant:

$$\mathcal{W} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

We are ready to obtain the derivatives of v_1 and v_2

$$\frac{dv_1}{dx} = \frac{1}{2} \begin{vmatrix} 0 & \sin 2x \\ x^2 & 2\cos 2x \end{vmatrix} = -\frac{1}{2}x^2 \sin 2x$$

$$\frac{dv_2}{dx} = \frac{1}{2} \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & x^2 \end{vmatrix} = \frac{1}{2}x^2 \cos 2x$$

Now that we have the formulas for the derivatives of v_1 and v_2, we have to do some antiderivatives to get v_1 and v_2.

$$v_1 = \int -\frac{1}{2}x^2 \sin 2x = \frac{1}{4}x^2 \cos 2x - \frac{1}{8}\cos 2x - \frac{1}{4}x \sin 2x + c_1$$

$$v_2 = \int \frac{1}{2}x^2 \cos 2x \, dx = \frac{1}{4}x^2 \sin 2x - \frac{1}{8}\sin 2x + \frac{1}{4}x \cos 2x + c_2$$

We are now ready to get our general solution:

$$\begin{aligned}
y &= v_1 \cos 2x + v_2 \sin 2x \\
&= \left(\frac{1}{4}x^2 \cos 2x - \frac{1}{8}\cos 2x - \frac{1}{4}x \sin 2x + c_1\right)\cos 2x \\
&\qquad\qquad + \left(\frac{1}{4}x^2 \sin 2x - \frac{1}{8}\sin 2x + \frac{1}{4}x \cos 2x + c_2\right)\sin 2x \\
&= c_1 \cos 2x + c_2 \sin 2x + \left(\frac{1}{4}x^2 - \frac{1}{8}\right)\left(\cos^2 2x + \sin^2 2x\right) \\
&= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x^2 - \frac{1}{8}
\end{aligned}$$

We have obtained the same solution that we had earlier when we used the method of undetermined coefficients, but variation of parameters caused us to do a lot more work. The main problem is that variation of parameters formula always requires us to do integrals to get v_1 and v_2. In the case of the above example, each integral required two integration by parts operations. By contrast, the method of undetermined coefficients requires us to take derivatives to get to our answer, but not integrals. Derivatives are generally easier than integrals. Therefore, given a choice between using the methods of undetermined coefficients or using variation of parameters, it's better to use undetermined coefficients. Unfortunately, we don't always have a choice. Consider the following example:

$$y'' + 4y = \sec 2x$$

We don't know the annihilator of $\sec 2x$ and we have no way to guess at the general form of the particular solution. Let's use variation of parameters to do this one. Just like the last example, we are looking for solutions of the form:

$$y = v_1 \cos 2x + v_2 \sin 2x$$

The Wronskian determinant $\mathcal{W}$ has the same value of 2 as the last example.

$$\frac{dv_1}{dx} = \frac{1}{2}\begin{vmatrix} 0 & \sin 2x \\ sec2x & 2\cos 2x \end{vmatrix} = -\frac{1}{2}\sin 2x \sec 2x$$

$$\frac{dv_2}{dx} = \frac{1}{2}\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sec 2x \end{vmatrix} = \frac{1}{2}\sec 2x \cos 2x = \frac{1}{2}$$

Now, it's time to integrate:

$$v_1 = \int -\frac{1}{2}\sin 2x \sec 2x \, dx = -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x}\, dx = \frac{1}{4}\ln|\cos 2x| + c_1$$
$$v_2 = \int \frac{1}{2}\, dx = \frac{1}{2}x + c_2$$

So our general solution is:

$$\begin{aligned} y &= v_1 \cos 2x + v_2 \sin 2x \\ &= \left(\frac{1}{4}\ln|\cos 2x| + c_1\right)\cos 2x + \left(\frac{1}{2}x + c_2\right)\sin 2x \\ &= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x \ln|\cos 2x| \end{aligned}$$

The particular solution is $y_p = \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x$ which is an expression we never would have guessed. Variation of parameters was the only way to do this problem.

Example:
Find the general solution of the following differential equation:

$$(D-1)^2 y = \frac{1}{x}e^x$$

The general homogeneous solution is a linear combination of $y_{1h} = e^x$ and $y_{2h} = xe^x$. Variation of parameters will give us two functions, v_1 and v_2 so that $y = v_1 e^x + v_2 x e^x$ will be the nonhomogeneous solution. The Wronskian determinant in this case is:

$$\mathcal{W} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2z} = e^{2x}$$

We can now use our formulas to get the derivatives of v_1 and v_2

$$\frac{dv_1}{dx} = e^{-2x} \begin{vmatrix} 0 & xe^x \\ \frac{1}{x}e^x & (x+1)e^x \end{vmatrix} = -1 \qquad \frac{dv_2}{dx} = e^{-2x} \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{x}e^x \end{vmatrix} = \frac{1}{x}$$

At least the integrals are easier this time.

$$v_1 = \int -1\, dx = -x + c_1 \qquad v_2 = \int \frac{1}{x}\, dx = \ln|x| + c_2$$

Now we can form our general nonhomogeneous solution:

$$y = v_1 e^x + v_2 x e^x = (-x + c_1)e^x + (\ln|x| + c_2)xe^x$$

This is the answer, but with a little rearrangement we can simplify it.

$$y = c_1 e^x + (c_2 - 1)xe^x + xe^x \ln|x|$$

If we let $a = c_1$ and $b = c_2 - 1$ we get:

$$y = ae^x + bxe^x + xe^x \ln|x|$$