vibration.* For $\gamma = \omega$ we define the solution to be

$$x(t) = \lim_{\gamma \to \omega} F_0 \frac{-\gamma \sin \omega t + \omega \sin \gamma t}{\omega(\omega^2 - \gamma^2)}$$

$$= F_0 \lim_{\gamma \to \omega} \frac{\dfrac{d}{d\gamma}(-\gamma \sin \omega t + \omega \sin \gamma t)}{\dfrac{d}{d\gamma}(\omega^3 - \omega\gamma^2)}$$

$$= F_0 \lim_{\gamma \to \omega} \frac{-\sin \omega t + \omega t \cos \gamma t}{-2\omega\gamma}$$

$$= F_0 \frac{-\sin \omega t + \omega t \cos \omega t}{-2\omega^2}$$

$$= \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t. \tag{9}$$

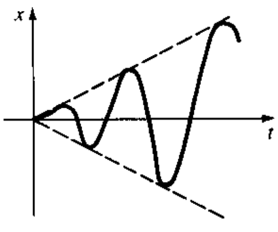

Figure 5.18

As suspected, when $t \to \infty$, the displacements become large; in fact, $|x(t_n)| \to \infty$, when $t_n = n\pi/\omega$, $n = 1, 2, \ldots$. The phenomenon we have just described is known as **pure resonance**. The graph given in Figure 5.18 shows typical motion in this case.

In conclusion it should be noted that there is no actual need to use a limiting process on (8) to obtain the solution for $\gamma = \omega$. Alternatively, equation (9) follows by solving the initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t$$

$$x(0) = 0, \qquad \left.\frac{dx}{dt}\right|_{t=0} = 0$$

directly by conventional methods.

Remark: If a mechanical system were actually described by a function such as (9) of this section, it would necessarily fail. Large oscillations of a weight on a spring would eventually force the spring beyond its elastic limit. One might argue too that the resonating model presented in Figure 5.18 is completely unrealistic since it ignores the retarding effects of ever-present damping forces. Although it is true that pure resonance cannot occur when the smallest amount of damping is taken into consideration, large and equally destructive amplitudes of vibration (although bounded as $t \to \infty$) can occur.

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* Forgetting about damping effects of shock absorbers, the situation is roughly equivalent to a number of passengers jumping up and down in the back of a bus in time with the natural vertical motion caused by equally spaced faults (such as cracks) in the road. Theoretically these passengers could upset the bus—assuming they are not kicked off first.

If you have ever looked out a window while in flight, you have probably observed that wings on an airplane are not perfectly rigid. A reasonable amount of flex or flutter is not only tolerated but necessary to prevent the wing from snapping like a piece of peppermint stick candy. In late 1959 and early 1960 two commercial plane crashes involving a relatively new model of propjet occurred, illustrating the destructive effects of large mechanical oscillations.

The unusual aspect of these crashes was that they both happened while the planes were in mid-flight. Barring midair collisions, the safest period during any flight is when the plane has attained its cruising altitude. It is well known that a plane is most vulnerable to an accident when it is least maneuverable, namely, during either take-off or landing. So, having two planes simply fall out of the sky was not only a tragedy but an embarrassment to the aircraft industry and a thoroughly puzzling problem to aerodynamic engineers. In crashes of this sort, a structural failure of some kind is immediately suspected. After a massive technical investigation, the problem was eventually traced in each case to an outboard engine and engine housing. Roughly, it was determined that when each plane surpassed a critical speed of approximately 400 mph, a propeller and engine began to wobble, causing a gyroscopic force, which could not be quelled or damped by the engine housing. This external vibrational force was then transferred to the already oscillating wing. This, in itself, need not have been destructively dangerous since aircraft wings are designed to withstand the stress of unusual and excessive forces. (In fact the particular wing in question was so incredibly strong that test engineers and pilots who were deliberately trying to snap a wing under every conceivable flight condition failed to do so.) But, unfortunately, after a short period of time during which the engine wobbled rapidly, the frequency of the impressed force actually slowed to a point at which it approached and finally coincided with the maximum frequency of wing flutter (around 3 cycles per second). The resulting resonance situation finally accomplished what the test engineers could not do; namely, the amplitudes of wing flutter became large enough to snap the wing (see Figure 5.19).

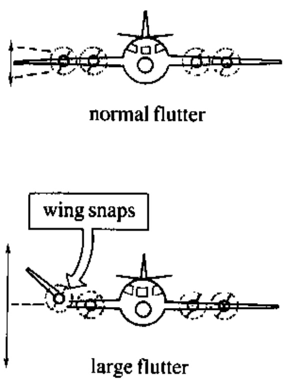

Figure 5.19

The problem was solved in two steps. All models of this particular plane were required to fly at speeds substantially below 400 mph until each plane could be modified by considerably strengthening (or stiffening) the engine housings. A strengthened engine housing was shown to be able to impart a damping effect capable of preventing the critical resonance phenomenon even in the unlikely event of a subsequent engine wobble.*

You may be aware that soldiers usually do not march in step across bridges. The reason for breaking stride is simply to avoid any possibility of resonance occurring between the natural vibrations inherent in the bridge's

* For a fascinating nontechnical account of the investigation see Robert J. Serling, *Loud and Clear*, New York: Dell, 1970, Chapter 5.

Figure 5.20 Courtesy of Wide World Photos/AP

structure and the frequency of the external force of a multitude of feet stomping in unison on the bridge.

Bridges are good examples of vibrating mechanical systems, which are constantly being subjected to external forces, from people walking on them, cars and trucks driving on them, water pushing against their foundations, and wind blowing against their superstructures. On November 7, 1940, the Tacoma Narrows Bridge at Puget Sound in the state of Washington collapsed. However, the crash came as no surprise since "Galloping Gertie," as the bridge was called by local residents, was famous for a vertical undulating motion of its roadway, which gave many motorists a very exciting crossing. On November 7, only four months after its grand opening, the amplitudes of these undulations became so large that the bridge failed and a substantial portion was sent splashing into the water below. In the investigation that followed, it was found that a poorly designed superstructure caused the wind blowing across it to vortex in a periodic manner. When the frequency of this periodic force approached the natural frequency of the bridge, large upheavals of the road resulted. In a word, the bridge was another victim of the destructive effect of mechanical resonance. Since this disaster developed over a matter of months, there was sufficient opportunity to record on film the strange and frightening phenomenon of a bucking and heaving bridge and its ultimate collapse (see Figure 5.20).*

Figure 5.21 © 1988 Memtek Products

Acoustic vibrations can be equally as destructive as large mechanical vibrations. In recent television commercials, jazz singers have inflicted destruction on the lowly wine glass (see Figure 5.21). Sounds from organs and piccolos have been known to crack windows.

> "As the horns blew, the people began to shout. When they heard the signal horn, they raised a tremendous shout. The wall collapsed. . . ." *Joshua* 6:20

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* National Committee for Fluid Mechanics Films, Educational Services, Inc., Watertown, Mass. See also, *American Society of Civil Engineers: Proceedings*, "Failure of the Tacoma Narrows Bridge," Vol. 69, pp. 1555–85, Dec. 1943.