$$(D^2 + \omega^2)x(t) = 0$$

If we substitute $x = e^{rt}$ we obtain:

$$r = \pm\omega i$$

Therefore:

$$x(t) = b_1 e^{i\omega t} + b_2 e^{-i\omega t}$$

“Divide 10 into two parts whose product is 40”

Jerome Cardan *Ars Magna - 1545*

Let x and y be the two parts we are dividing 10 into

$$x + y = 10 \qquad\qquad xy = 40$$

Substitute $y = 10 - x$ into $xy = 40$

$$x(10 - x) = 40$$

$$10x - x^2 = 40$$

$$0 = x^2 - 10x + 40$$

$$x^2 - 10x = -40$$

Add 25 to both sides

$$x^2 - 10x + 25 = -15$$

$$(x - 5)^2 = -15$$

$$x - 5 = \pm\sqrt{-15}$$

$$x = 5 \pm \sqrt{-15}$$

If we add $5 - \sqrt{-15}$ to $5 + \sqrt{-15}$ we get 10

If we multiply $5 - \sqrt{-15}$ by $5 + \sqrt{-15}$ we get 40

$$
\begin{aligned}
(5 - \sqrt{-15})(5 + \sqrt{-15}) &= 5^2 - (\sqrt{-15})^2 \\
&= 25 - (-15) \\
&= 40
\end{aligned}
$$

$$i = \sqrt{-1}$$

Rene' Descartes (1628)

$$5 + \sqrt{-15} = 5 + \sqrt{(-1)(15)} = 5 + i\sqrt{15}$$

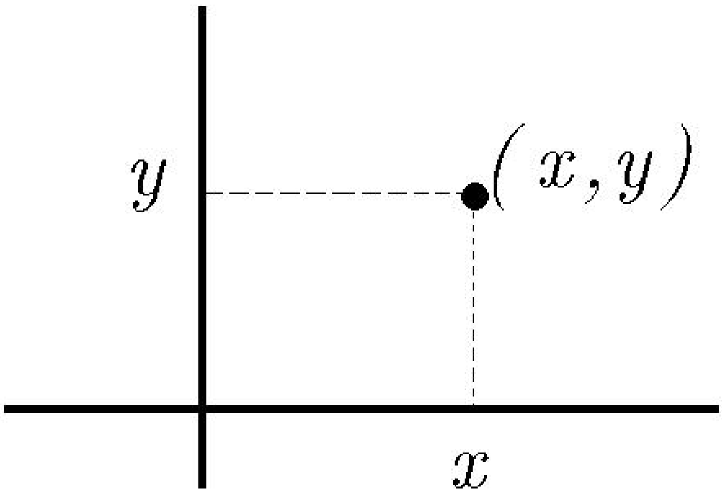
Addition of complex numbers
If $z_1 = a + bi$ and $z_2 = c + di$ then

$$z_1 + z_2 = a + c + (b + d)i$$

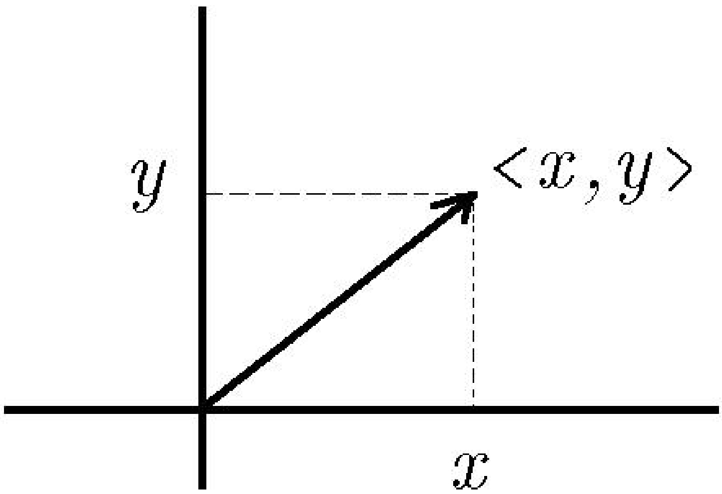
Multiplication of complex numbers

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + (bi)(di) \\ &= ac - bd + (ad + bc)i\end{aligned}$$

John Wallis (1685) represented $x + yi$ as a point in a plane



This can also be depicted as a vector



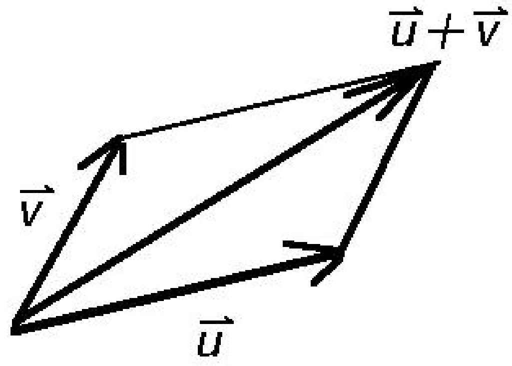
William Rowan Hamilton (1830)

The notation $a + bi$ is just another notation for the vector $\langle a, b \rangle$

Addition of complex numbers is just vector addition

$$(a + bi) + (c + di) = a + c + (b + d)i$$

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c,\ b + d\rangle$$

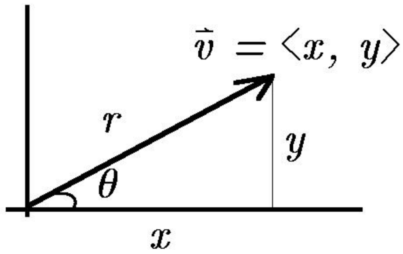


The only new thing is a new "multiplication" of vectors called complex multiplication

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

$$\langle a, b\rangle\langle c, d\rangle = \langle ac - bd,\ ad + bc\rangle$$

We may use polar coordinates to describe vectors.



$$z = x + iy$$

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$z = x + iy$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

A special case would be a unit vector

$$z = \cos\theta + i\sin\theta$$

$$z = \cos\theta + i\sin\theta$$

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

$$z = \cos\theta + i\sin\theta$$

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

$$\begin{aligned}\frac{dz}{d\theta} &= i^2\sin\theta + i\cos\theta \\ &= i(i\sin\theta + \cos\theta) \\ &= iz\end{aligned}$$

$$z = \cos\theta + i\sin\theta$$

$$\frac{dz}{d\theta} = iz \qquad \text{where } z(0) = 1$$

$$\frac{dz}{d\theta} = iz \qquad \text{where } z(0) = 1$$

$$\frac{dz}{z} = i\,d\theta$$

$$\int \frac{dz}{z} = i \int d\theta$$

$$\log_e z = i\theta + C$$

$$z(0) = 1 \text{ implies } C = 0$$

$$\log_e z = i\theta$$

$$z = e^{i\theta}$$

$$\log_e z = i\theta$$

$$z = e^{i\theta}$$

We started with $z = \cos\theta + i\sin\theta$ so:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's Formula

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)
\end{aligned}
$$

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}
$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Replace θ with $-\theta$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

Substitute $y = e^{rx}$

$$\frac{d^2y}{dx^2} + 4y = 0$$

Substitute $y = e^{rx}$

$$r^2 e^{rx} + 4e^{rx} = 0$$

$$r^2 + 4 = 0$$

$$r^2 = -4$$

$$r = \pm 2i$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

e^{2ix} and e^{-2ix} are solutions so the general solution is:

$$y = c_1 e^{2ix} + c_2 e^{-2ix}$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

e^{2ix} and e^{-2ix} are solutions so the general solution is:

$$\begin{aligned} y &= c_1 e^{2ix} + c_2 e^{-2ix} \\ &= c_1(\cos 2x + i \sin 2x) + c_2(\cos 2x - i \sin 2x) \\ &= (c_1 + c_2) \cos 2x + (ic_1 - ic_2) \sin 2x \end{aligned}$$

$$\begin{aligned} y &= c_1 e^{2ix} + c_2 e^{-2ix} \\ &= c_1(\cos 2x + i \sin 2x) + c_2(\cos 2x - i \sin 2x) \\ &= (c_1 + c_2) \cos 2x + (ic_1 - ic_2) \sin 2x \\ &= a \cos 2x + b \sin 2x \end{aligned}$$

where $a = c_1 + c_2$ and $b = ic_1 - ic_2$

Example.
Solve the following differential equation along with the given initial conditions:

$$y'' + 4y' + 5y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

Begin by substituting $y = e^{rx}$

$$\left(D^2 + 4D + 5\right)\left(e^{rx}\right) = 0$$

$$r^2 e^{rx} + 4re^{rx} + 5e^{rx} = 0$$

$$r^2 + 4r + 5 = 0$$

We could solve this with the quadratic formula or by completing the square.

$$r^2 + 4r + 5 = 0$$
$$r^2 + 4r + 4 = -1$$
$$(r + 2)^2 = -1$$
$$r + 2 = \pm i$$
$$r = -2 \pm i$$

If $r = -2 + i$ then $e^{(-2+i)x}$ is a solution.
If $r = -2 - i$ then $e^{(-2-i)x}$ is a solution.
The general solution is:

$$y = c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x}$$

If $r = -2 + i$ then $e^{(-2+i)x}$ is a solution.
If $r = -2 - i$ then $e^{(-2-i)x}$ is a solution.
The general solution is:

$$\begin{aligned} y &= c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x} \\ &= e^{-2x} \left(c_1 e^{ix} + c_2 e^{-ix} \right) \end{aligned}$$

$$
\begin{aligned}
y &= c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x} \\
&= e^{-2x}\left(c_1 e^{ix} + c_2 e^{-ix}\right) \\
&= e^{-2x}\left(c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)\right) \\
&= e^{-2x}\left((c_1 + c_2)\cos x + (c_1 i - c_2 i)\sin x\right) \\
&= e^{-2x}\left(a \cos x + b \sin x\right) \\
&= a e^{-2x}\cos x + b e^{-2x}\sin x
\end{aligned}
$$

In general, if we substitute e^{rx} into the differential equation and we obtain the complex roots:

$$r = \alpha \pm i\beta$$

then the following will be solutions of the differential equation:

$$e^{\alpha x} \cos \beta x \qquad e^{\alpha x} \sin \beta x$$

$$y'' + 4y' + 5y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$$y = ae^{-2x}\cos x + be^{-2x}\sin x$$

What about the initial conditions?

$$0 = y(0) = ae^{0}\cos 0 + be^{0}\sin 0 = a$$

If $a = 0$ then,

$$y = be^{-2x}\sin x$$

Next, impose the initial condition $y'(0) = 1$

$$y = be^{-2x} \sin x$$

$$\begin{aligned} y' &= D\left(be^{-2x} \sin x\right) \\ &= be^{-2x}(D-2)(\sin x) \\ &= be^{-2x}(\cos x - 2\sin x) \end{aligned}$$

$$1 = y'(0) = be^{0}(\cos 0 - 2\sin 0) = b$$

$$y = e^{-2x} \sin x$$