MA 345 Differential Equations - Assignments **C. Jacobs**

This assignment sheet is coordinated with Edition 9 of the Nagle, Saff and Snider text

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This assignment sheet was modified on March 13, 2020. Some problems have been deleted.

Assignment 0. *Integration Review*

Go back to your calculus textbook and review the chapter on integration. This is a review assignment and it will not be collected or graded.

(1) $$\int x^2 \ln x \, dx$$

(2) $$\int x \cosh x \, dx$$

(3) $$\int \sqrt{4 - x^2} \, dx$$

(4) $$\int \frac{x+4}{x+2} \, dx$$

(5) $$\int \frac{8x^2}{(x^2+1)(x-1)} \, dx$$

Assignment 1. *Introduction to Differential Equations. Separation of Variables*

Read 1.1, 1.2, 2.1 and 2.2

You should be able to do the following problems:

Exercise 1.2 Problems 1 - 8, Exercise 2.2 Problems 7 - 26

Hand in the following problems:

Solve each of the following differential equations. For problems 1 - 3, you are given no initial conditions and your solution will be a general solution. For problems 4 and 5, you are given initial conditions which will give specific values to the constants of integration that appear in the answer.

1. $$\frac{dy}{dx} = 1 + y^2$$

2. $$\frac{dy}{dx} + 7y = 0$$

3. $$\frac{dy}{dx} = \frac{y(y+1)}{x}$$

4. $$y \frac{dy}{dx} = x + xy^2 \qquad y(0) = 2$$

5. $$\frac{dy}{dx} + xy = x \qquad y(0) = 2$$

Assignment 2. *Applications of First Order*

Read 3.1 and 3.2

You should be able to do the following problems:

Exercise 3.2 Problems 1 - 25

Hand in the following problems:

1. Blood carries a drug into an organ at a rate of 3 cm^3/sec and leaves at the same rate. The organ has a liquid volume of 150 cm^3. The concentration of the drug in the blood entering the organ is $\frac{1}{3}$ gm/cm^3. What is the mass of the drug in the organ at time t if there was no trace of the drug initially?

2. A 1 liter tank is filled with a salt solution containing 4 grams of dissolved salt. At $t = 0$, a hole is drilled in the bottom of the tank and the solution drains out at $\frac{1}{3}$ liters per minute. At the same time, *fresh* water is being pumped in at $\frac{1}{3}$ liters per minute.

Let $x(t)$ be the amount of salt in grams after t minutes.

a) Find a formula for $x(t)$ by solving the appropriate differential equation.

b) How long will it take until 1 gram of salt is left in the tank?

3. At $t = 0$, 1 mole of chemical A and 1 mole of chemical B are dissolved in a liter of water. Eventually, these chemicals will combine to produce 1 mole of chemical C. However, the reaction does not take place immediately. At any point in time, the rate at which chemical C is produced is proportional to the product of the remaining number of moles of chemical A times the remaining number of moles of chemical B.

Assume that at $t = 0$, there is no chemical C present at all, but by $t = 1$ minute, a half mole of chemical C has been produced.

Let $x(t)$ be the number of moles of chemical C in the solution after t minutes.

Write an appropriate differential that determines $x(t)$ and solve it. Determine all constants present in your solution.

4. A cup of coffee is heated to 100 degrees Centigrade at $t = 0$ minutes. The cup is placed in a refrigerator which is kept constant at 0 degrees Centigrade. Let $u(t)$ denote the temperature of the coffee after t minutes.

a) Obtain the correct expression for $u(t)$ by solving an appropriate differential equation.

b) If the temperature of the coffee is 80 degrees after 1 minute, find out how long it will take for the coffee to cool down to 10 degrees.

5. An object of 5 kg is released from rest 1000 meters above the ground level and allowed to fall under the influence of gravity. Assuming that the force due to air resistance is proportional to the velocity of the object with proportionality constant $k = 50$ kg/sec, determine the formula for the velocity of the object.

Assignment 3. *Exact Differential Equations*

Read 2.3 and 2.4
You should be able to do the following problems:
Exercises 2.4 Problems 9 - 26

Hand in the following problems:

Find the solution of each of the following equations.

1. $$(e^x + 2xe^y)\,dx + x^2e^y\,dy = 0$$

2. $$(\sin y - \sin x)\,dx + (1+x)\cos y\,dy = 0 \qquad y(0) = \pi$$

3. $$\left(\sqrt{y} - 2x\right)\,dx + \left(\frac{x+1}{2\sqrt{y}}\right)\,dy = 0 \qquad y(0) = 1$$

4. $$(2x + y)\,dx = (2y - x)\,dy$$

5. $$\frac{dy}{dx} = \frac{1 - 2x + 2y}{2y - 2x}$$

Assignment 4. *First Order Linear Differential Equations*

Read 2.3

You should be able to do the following problems:

Exercise 2.3 Problems 7 - 25

Hand in the following problems:

Find the solutions of each of the following equations

1. $$\frac{dy}{dx} - 2y = e^x \qquad y(0) = 0$$

2. $$\frac{dy}{dx} = x + y \qquad y(0) = 2$$

3. $$x\frac{dy}{dx} = \tan x - y$$

4. $$\left(x + x^2\right)\frac{dy}{dx} + xy = 1 + x$$

5. $$\frac{dy}{dx} - \frac{2y}{x} = 1$$

Assignment 5. *Special Integrating Factors, Substitutions and Transformations*

Read 2.5 and 2.6

You should be able to do the following problems:

Exercise 2.5 Problems 7 - 12, Exercise 2.6 Problems 9 - 16

Hand in the following problems:

Find the solutions of each of the following equations.

1. $$x^2\frac{dy}{dx} = y^2 + xy \qquad \textit{Hint: Try } v = \frac{y}{x}$$

2. $$(e^x + y)\,dx + x\,dy = 0$$

3. $$\left(e^{2x} - y^3\right)\,dx + 3y^2\,dy = 0 \qquad y(0) = 0$$

4. $$(2xy - 1)\,dx + \frac{x}{y}\,dy = 0$$

5. $$\left(\frac{1}{y} + x\right)\,dx = \frac{1}{y^2}\,dy$$

Assignment 6. *Matrix Reduction*

Use the method of **matrix reduction** (demonstrated in class) to solve the following systems of equations. Indicate explicitly whenever a particular system of equations has no solution or has infinitely many solutions. Show all work.

(1)
$$\begin{aligned} 2x - y &= -1 \\ -x + y &= 1 \end{aligned}$$

(2)
$$\begin{array}{ccccc} x & + & y & = & 0 \\ 2x & + & 3y & = & 0 \end{array}$$

(3)
$$\begin{array}{ccccccc} x & + & 3y & + & 3z & = & -1 \\ x & + & y & & & = & 1 \\ x & + & 2y & + & z & = & 0 \end{array}$$

(4)
$$\begin{array}{ccccccc} x & + & y & - & z & = & 0 \\ 2x & + & y & + & 2z & = & 0 \\ 5x & + & 4y & - & z & = & 0 \end{array}$$

(5)
$$\begin{aligned} x + y &= 0 \\ x + z &= 0 \\ 2x + y + z &= 4 \end{aligned}$$

Assignment 7. *Matrix Algebra*

Read 9.2, 9.3 You should be able to do the following problems:
Exercise 9.2 Problems 1 - 9 and 9.3 Problems 1 - 8

Hand in the following problems: **1.** Let's define matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{I}$ as follows:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Calculate $\mathbf{A} + \mathbf{B}$ and $4\mathbf{I} - \mathbf{A}$

2. Let $\mathbf{X} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{I}$ be defined exactly as in problem 1. Calculate each of the following matrix operations. If the operation is not defined, then state this explicitly.

$$\mathbf{AX} \qquad \mathbf{XI} \qquad \mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} \qquad \mathbf{AB}$$

3. Let $\mathbf{A}$ and $\mathbf{X}$ be defined exactly as in problems 1 and 2 and let $\mathbf{V} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. Find the vector $\mathbf{X}$ that solves the equation $\mathbf{AX} = \mathbf{V}$.

4. Define $\mathbf{A}$, $\mathbf{X}$, $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -3 \\ -1 & -2 & 3 \\ 1 & 1 & -2 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{E}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{E}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{E}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Solve each of the following equations:

$$\mathbf{AX} = \mathbf{E}_1 \qquad \mathbf{AX} = \mathbf{E}_2 \qquad \mathbf{AX} = \mathbf{E}_3$$

Call the solutions of these equations $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ respectively. Form a three by three matrix $\mathbf{B}$ where the first column is $\mathbf{B}_1$, the second column is $\mathbf{B}_2$ and the third column is $\mathbf{B}_3$. Then, calculate the product $\mathbf{AB}$.

5. Define $\mathbf{P}$, $\mathbf{X}$, $\mathbf{O}$ and $\mathbf{I}$ as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solve each of the following equations:

$$\mathbf{PX} = \mathbf{O} \qquad (\mathbf{P} - 2\mathbf{I})\mathbf{X} = \mathbf{O}$$

Assignment 8. *Matrix Algebra. Inverses and Determinants*
Read 9.1, 9.2, 9.3
You should be able to do the following problems:
Exercise 9.3 Problems 9 - 16
Hand in the following problems:

Hand in the following problems:

1. For each of the following matrices, determine whether or not the inverse exists. If the inverse exists, find it. Show all work. If you think you have a matrix B that is the inverse of A, check your work by verifying that $AB = I$.

$$(a) \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad (b) \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(c) \quad \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \end{pmatrix} \qquad (d) \quad \begin{pmatrix} 2 & 4 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \qquad (e) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

2. Find the inverse of the matrix:

$$\begin{pmatrix} 3 & -6 & 4 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

and use this inverse to solve the following system of linear equations:

$$\begin{aligned} 3x - \ \ 6y + 4z &= \ 1 \\ -x + 3y - 2z &= \ 0 \\ -y + \ \ z &= \ 1 \end{aligned}$$

3. Solve the following system of equations using *Cramer's Rule.*

$$\begin{aligned} 2x + 3y + 3z &= 1 \\ 3x + 5y + 5z &= -1 \\ x + 2y + 3z &= 1 \end{aligned}$$

4. Find all values of λ for which the following matrix has no inverse:

$$\begin{pmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{pmatrix}$$

5. Define $\mathbf{A}$, $\mathbf{I}$, $\mathbf{O}$ and $\mathbf{X}$ as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The equation $\mathbf{AX} = \lambda\mathbf{X}$ can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{O}$$

Find all solutions $\mathbf{X}$ for each of the following cases:

$$\lambda = 0 \qquad \lambda = 1 \qquad \lambda = -1$$

Assignment 9. *Linear Differential Equations - Distinct Root Case*

Read 4.1 - 4.2
You should be able to do the following problems:
Exercise 4.2 Problems 1 - 20

Hand in the following problems:

Solve each of the following:

Find the solution of each of the following equations

(1)
$$y'' + 2y' - 7y = 0$$

(2)
$$4y'' - 4y' - 3y = 0$$

(3)
$$(D^3 + 3D^2 - 4D)y = 0$$

(4)
$$y'' + 5y' = 14y$$

(5)
$$\frac{d^2y}{dx^2} = 4\,\frac{dy}{dx} \quad \text{where} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 4$$

Assignment 10. *The Exponential Shift Theorem*

Read the notes on the Exponential Shift Theorem that are posted on the course website.
Hand in the following problems:

Problems 1 - 4. Use the Exponential Shift Theorem to calculate the following expressions:

1.
$$\left(D^2 - 4D + 5\right)\left(e^{2x}\sin x\right)$$

2.
$$\left(D^2 - 1\right)\left(e^x + xe^x + x^2e^x\right)$$

3.
$$D^3\left(e^{-x}x\right)$$

4.
$$(D-4)^2\left(e^{4x}x^4\right)$$

5. Solve the following differential equation by making the substitution $u = e^{-x}y$ and using the Exponential Shift Theorem.
$$(D-1)^2y = 0$$

Assignment 11. *Repeated and Complex Root Cases*

Read 4.3, 6.2

You should be able to do the following problems:

Exercise 4.3 Problems 1 - 27, Exercise 6.2 Problems 1 - 6, 15 - 18

Hand in the following problems:

1.
$$(4D^2 + 4D + 1)y = 0$$

2.
$$(D + 2)(D^2 + D - 2)y = 0$$

3.
$$(D^2 - 16)(D - 4)y = 0$$

4.
$$(D^2 - 6D + 10)y = 0$$

5.
$$y'' + y' = y$$

6.
$$(D^4 + 4D^2 + 4)y = 0$$

7.
$$(D - 2)(D^2 - 2D)y = 0$$

8.
$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 9x = 0 \qquad x(0) = 0 \qquad x'(0) = 1$$

9. Suppose a circuit has an inductor of inductance $L = 1$, a resistor of resistance $R = 4$ and a capacitor of capacitance $C = \frac{1}{2}$, all connected in series. If $Q(t)$ is the electric charge at time t, then the differential equation that determines $Q(t)$ is:

$$\frac{d^2Q}{dt^2} + 4\frac{dQ}{dt} + 2Q = 0$$

Find the general solution of this equation.

10. Find the solution of the following differential equation.

$$y'' + 2y = 0$$

Assume that the solution satisfies the initial conditions $y(0) = 0$ and $y'(0) = 2$

Assignment 12. *Introduction to Nonhomogeneous Equations*

Read 4.4 Hand in the following problems:

If L is a linear operator, an equation of the form $Ly = 0$ is called a *homogeneous equation* and an equation of the form $Ly = f$ is called a *nonhomogeneous equation*. The solution of $Ly = 0$ is related to the solution of $Ly = f$. When you solve each of the following equations, look for the relationship between these solutions.

1a.
$$(D - 2)y = 0$$

1b.
$$(D - 2)y = e^x$$

2a.
$$\frac{dy}{dx} - xy = 0$$

2b.
$$\frac{dy}{dx} - xy = x$$

3. You can reduce the following second order equations to first order equations by making the substitution $u = y'$. Solve each of the following.

3a.
$$y'' - y' = 0$$

3b.
$$y'' - y' = e^x$$

4. Define $\mathbf{A}$, $\vec{\mathbf{x}}$, $\vec{\mathbf{0}}$ and $\vec{\mathbf{b}}$ as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ -2 & 2 & 0 \end{pmatrix} \qquad \vec{\mathbf{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \vec{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\mathbf{b}} = \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$$

Use matrix reduction (or any legitimate method) to solve each of the following matrix equations:

4a.
$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

4b.
$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

Assignment 13. *Method of Undetermined Coefficients. Annihilators*

Read 4.4, 4.5, 6.3

You should be able to do the following problems:

Exercise 4.4 Problems 9 - 36, Exercise 4.5 Problems 3 - 40

Exercise 6.3 Problems 1 - 30

Hand in the following problems:

1. Find the annihilators for each of the following functions:

a. $x + e^{2x}$ **b.** xe^{2x} **c.** $e^x + e^{-x}$ **d.** $2\sin 2x + 4\cos 2x$

2.
$$\frac{d^2y}{dx^2} + y = x^2 + x$$

3.
$$(D-3)^2 y = e^{2x}$$

4.
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 2e^x + 4e^{-x}$$

5.
$$(D^2 - 2D)y = 8e^{2x}$$

Assignment 14. *Applications to Springs and Circuits*

Read 4.9, 4.10, 5.6 and 5.7

You should be able to do the following problems:

Exercise 4.9 Problems 1 - 10, Exercise 4.10 Problems 8 - 15

Hand in the following problems:

1. A spring with a spring constant of $k = 16$ N/m (Newtons per meter) is attached to a mass of $m = 2$ kg. Let x be the displacement of the mass relative to the equilibrium position. Suppose the initial displacement is $x(0) = 4$ m and the initial velocity is $v(0) = 0$. Write a differential equation that determines x as a function of t. Solve the differential equation.

2. An object of mass 4 kg is attached to a spring with spring constant $k = 100$ N/m. Assume a resistive force due to friction with a damping constant of $b = 32$ kg/sec. If the object begins at the equilibrium position with an initial velocity of 3 m/sec, determine the position of the object as a function of time by solving an appropriate differential equation.

3. An object of mass 1 kg is attached to a spring with spring constant $k = 4$ N/m. There is a resistive force due to friction with a damping constant of $b = 4$ kg/sec. There is also a time dependent force acting on the object of $F(t) = 3te^{-2t}$. Let $x(t)$ denote the position of the object relative to the equilibrium position after t seconds. Assume that at $t = 0$, the object begins at its equilibrium position with a velocity of 0. Set up and solve the differential equation that determines $x(t)$.

4. An object of mass 2 kilograms is attached to a spring with a spring constant $k = 8$ newtons/meter. Assume that there is no damping force due to friction, but there is a time dependent force of $F(t) = 12\cos 2t$. Assume also that the object begins with an initial velocity of 0 and an initial displacement of 0. Determine the displacement of the object as a function of time by solving an appropriate differential equation.

5. An electric circuit has an electromotive force given by $\mathcal{E}(t) = 60\cos 5t$ volts, an inductor of 1 henry, a resistor of 6 ohms and a capacitor of 0.04 farads. Let $Q(t)$ be the charge on the capacitor. $Q(t)$ will solve the differential equation:

$$\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 25Q = 60\cos 5t$$

If $Q(t)$ satisfies the initial conditions $Q(0) = 0$ and $Q'(0) = 6$, find the formula for $Q(t)$.

Assignment 15. *Variation of Parameters*

Read 4.6, 6.4
You should be able to do the following problems:
Exercise 4.6 Problems 1 - 18, Exercise 6.4 Problems 1 - 6
Hand in the following problems:

Solve each of the following by variation of parameters.

1.

$$y'' + y = \sin x \cos x$$

2.

$$(D^2 - D)y = \frac{e^{2x}}{e^{2x} + 1}$$

3.

$$y'' - 2y' + y = x^2 e^x$$

4. The Method of Variation of Parameters can be used to find the general solution of the following differential equation:

$$\left(D - \frac{1}{2}\right)^2 y = x^{-2} e^{\frac{x}{2}}$$

The solution has the form:

$$y = v_1(x) \cdot e^{\frac{x}{2}} + v_2(x) \cdot x e^{\frac{x}{2}}$$

where $v_1(x)$ and $v_2(x)$ are functions. Calculate the function $v_1(x)$

Assignment 16. *Introduction to Laplace Transforms*

Read 7.1, 7.2, 7.3, 7.4, 7.5
You should be able to do the following problems:
Exercise 7.2 Problems 1 - 20, Exercise 7.3 Problems 1 - 25
Exercise 7.4 Problems 1 - 30
Hand in the following problems:

1. The Laplace transform of a function $f(t)$ is defined to be: $\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$. Use this integral to calculate the Laplace transforms of each of the following functions:

a) $t+1$ b) $(t+1)e^{-4t}$

2. Use a table of Laplace transforms as well as any relevant properties of Laplace transforms to find the following transforms:

a) $\mathcal{L}\left(e^{6t} - e^{-6t}\right)$ b) $\mathcal{L}\left(t^2 e^{-t}\right)$ c) $\mathcal{L}(\cos 2t)$ d) $\mathcal{L}\left(e^{3t}\cos 2t\right)$ e) $\mathcal{L}\left(e^{-t}\cosh t\right)$

3. Calculate the Laplace transform of the following function:

$$f(t) = \begin{cases} te^t & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t \end{cases}$$

4. Calculate the inverse Laplace transforms of each of the following:

a) $\frac{6}{(s-2)^2}$ b) $\frac{1}{(2s+1)^2}$ c) $\frac{8}{(s+2)(s+4)}$ d) $\frac{3}{(s^2+1)(s^2+4)}$ e) $\frac{1}{s^4+s^3}$

Assignment 17. *Using Laplace Transforms to Solve Differential Equations*

Read 7.5
You should be able to do the following problems:
Exercise 7.5 Problems 1 - 32
Hand in the following problems:

Problem 1 has been deleted

For problems 2 - 5, use the method of *Laplace transforms* to solve the differential equations:

2. $$y'' - 4y = -6e^t, \qquad y(0) = 3 \; y'(0) = 0$$

3. $$y'' + 2y' = 2t + 3, \qquad y(0) = 0, \; y'(0) = 1$$

4. $$y'' + 8y' + 25y = 100, \qquad y(0) = 2, \; y'(0) = 20$$

5. $$y'' - 4y' + 4y = t^3 e^{2t} \qquad y(0) = 0, \; y'(0) = 0$$

Assignment 18. *Additional Laplace Transform Topics - Step Functions*

Read 7.6, 7.7, 7.8 and 7.9

You should be able to do the following problems:

Exercise 7.6 Problems 1 - 36, Exercise 7.8 Problems 1 - 12, Exercise 7.9 Problems 13 - 20

Hand in the following problems:

1. Graph each of the following functions. The symbol $\mathcal{U}$ refers to the *unit step function.*

a. $f(t) = (\cos t) \cdot (\mathcal{U}(t) - \mathcal{U}(t - \pi))$
b. $g(t) = \mathcal{U}(t) - \mathcal{U}(t - 1) + (2 - t) \cdot (\mathcal{U}(t - 1) - \mathcal{U}(t - 2))$

2. Express the following $f(t)$ in terms of the unit step function $\mathcal{U}$ and then calculate $\mathcal{L}(f)$. You may use a table of Laplace transforms.

$$f(t) = \begin{cases} t & \text{when } 0 \le t < 1 \\ e^{1-t} & \text{when } t \ge 1 \end{cases}$$

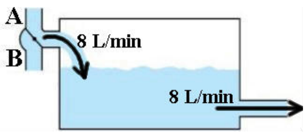
3. The electric charge $y(t)$ in an LC series circuit is governed by the initial value problem:

$$y''(t) + 4y(t) = 3\sin t - 3\sin t \cdot \mathcal{U}(t - 2\pi), \qquad y(0) = 1, \;\; y'(0) = 3$$

where $\mathcal{U}$ is the *unit step function.* Determine the charge as a function of time t.

Problem 4 has been deleted

5. At $t = 0$, a tank contains 16 liters of a brine solution with 10 grams of salt dissolved in it. Then, we start draining fluid out at 8 liters per minute. For $0 \le t \le 2$, valve A is open and only pure water is being pumped in at 8 liters per minute. Then, for $t > 2$, valve A is closed and valve B is opened and salt solution containing $\frac{1}{2}$ grams of salt per liter is pumped in at 8 liters per minute.



a. Let $x(t)$ be the number of grams of salt in the tank after t minutes. Set up an appropriate differential equation for $x(t)$, making appropriate use of the unit step function $\mathcal{U}$.

b. Solve the differential equation using the method of Laplace transforms. Your final answer should be expressed in terms of the unit step function.

Assignment 19. *Series Solutions*

Read 8.2 and 8.3
You should be able to do the following problems:
Exercise 8.3 Problems 11 - 28
Hand in the following problems:

Problems 1 - 3 are to be solved by the method of *infinite series.* This means that you must obtain a solution of the form $\sum a_n x^n$. You must solve for a_n so that the sum solves the given differential equation. Try to express your final answer in summation notation.

1. $y'' - y = 0$ where $y(0) = y'(0) = 1$

2. $y'' - 2xy' - 2y = 0$ where $y(0) = 1,\ y'(0) = 0$

3. $(1 - 2x)y'' - 4y' = 0$ where $y(0) = 0,\ y'(0) = 2$

4. The following differential equation has a solution of the form $\sum a_n x^n$. Calculate only the coefficients a_2, a_3 and a_4.

$$\left(1 + x^2\right) y'' - 2y = 0 \quad \text{where } \; y(0) = y'(0) = 1$$

Assignment 20. *Eigenvalues, Eigenvectors and Matrix Differential Equations*

Read 9.3, 9.4, 9.5, 9.6

You should be able to do Exercises 9.5 Problems 1 - 16

Hand in the following problems:

(1) For each of the following matrices, calculate all eigenvalues and corresponding eigenvectors.

$$(a)\quad \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \qquad\qquad (b)\quad \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}$$

$$(c)\quad \begin{pmatrix} 3 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad\qquad (d)\quad \begin{pmatrix} 0 & 1 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & -3 \end{pmatrix}$$

(2) Find the general solution of the differential equation

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \qquad \text{where} \quad A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The coordinates x_1 and x_2 are functions of t.

(3) Find the general solution of the differential equation

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \qquad \text{where} \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The coordinates x_1 and x_2 are functions of t.

(4) Find the solution of the differential equation

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \qquad \vec{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{where} \quad A = \begin{pmatrix} -7 & 24 \\ -2 & 7 \end{pmatrix} \quad \text{and} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The coordinates x_1 and x_2 are functions of t.

Problem 5 has been deleted